Hajłasz-Sobolev Imbedding and Extension

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Abstract   The author establishes some geometric criteria for a Hajłasz-Sobolev $\dot{M}_{\text{ball}}^{s,p}$-extension (resp. $\dot{M}_{\text{ball}}^{s,p}$-imbedding) domain of $\mathbb{R}^n$ with $n \geq 2$, $s \in (0,1]$ and $p \in [n/s, \infty]$ (resp. $p \in (n/s, \infty]$). In particular, the author proves that a bounded finitely connected planar domain Ω is a weak α-cigar domain with $\alpha \in (0,1)$ if and only if $\dot{F}_{p,\infty}^s(\mathbb{R}^2)|_\Omega = \dot{M}_{\text{ball}}^{s,p}(\Omega)$ for some/all $s \in [\alpha, 1)$ and $p = (2-\alpha)/(s-\alpha)$, where $\dot{F}_{p,\infty}^s(\mathbb{R}^2)|_\Omega$ denotes the restriction of the Triebel-Lizorkin space $\dot{F}_{p,\infty}^s(\mathbb{R}^2)$ on Ω.

1   Introduction

Let $X(\Omega)$ and $Y(\Omega)$ be function spaces defined on a domain $\Omega \subset \mathbb{R}^n$. Then Ω is called an X-extension domain if $X(\Omega) = X(\mathbb{R}^n)|_\Omega$ with equivalent norms, where $X(\mathbb{R}^n)|_\Omega \equiv \{u|_\Omega : u \in X(\mathbb{R}^n)\}$ and for $v \in X(\mathbb{R}^n)|_\Omega$, $\|v\|_{X(\mathbb{R}^n)|_\Omega} \equiv \inf \|u\|_{X(\mathbb{R}^n)}$ with the infimum taken over all $u \in X(\mathbb{R}^n)$ such that $u|_\Omega = v$. Also Ω is said to support an imbedding from $X(\Omega)$ to $Y(\Omega)$ if $X(\Omega)$ is a subset of $Y(\Omega)$ and for all $u \in X(\Omega)$, $\|u\|_{Y(\Omega)} \leq C\|u\|_{X(\Omega)}$ with constant C independent of u. Moreover, we always denote by $\dot{W}^{1,p}(\Omega)$ (resp. $W^{1,p}(\Omega)$) with $p \in (1, \infty]$ the homogeneous (resp. inhomogeneous) Sobolev space. The other notions for domains, such as uniform domain, weak α-cigar domain, regular domain, the LLC property and the slice property, will be explained in Section 2.

For the history of geometric properties of Sobolev extension and imbedding domains see, for example, [9, 17, 24, 5, 6, 18, 1, 2, 19, 3, 30, 31, 13, 14] and their references. In particular, it was proved by Jones [17] that a bounded simply connected domain $\Omega \subset \mathbb{R}^2$ is a uniform domain if and only if it is a $W^{1,2}$-extension domain; and by Gehring and Martio [6] that the $W^{1,n}$-extension domain has the LLC property; see also [18, 7, 8, 9, 36]. On the other hand, Gehring and Martio [5] proved that for $\alpha \in (0,1]$, Ω is a weak α-cigar domain if and only if it is a local Lip_α-extension domain. Moreover, let $p \in (n, \infty)$ and $\alpha = (p-n)/(p-1)$. It was proved by Buckley and Koskela [2] that a weak α-cigar domain always supports an imbedding from $\dot{W}^{1,q}(\Omega)$ into $\dot{C}^{1-n/q}(\overline{\Omega})$ for all $q \in [p, \infty)$; and by Koskela [19] that a weak α-cigar domain is a $W^{1,q}$-extension domain for all $q \in (p, \infty)$, which was further improved by Shvartsman [31] to all $q \in (p^*, \infty)$ with some $p^* \in (n, p)$. Conversely, with the additional assumption that Ω has the slice property, Buckley and

Koskela [2] proved that if Ω supports an imbedding from $\dot{W}^{1,p}(\Omega)$ into $\dot{C}^{1-n/p}(\overline{\Omega})$, then it is a weak α-cigar domain. We also refer the reader to [3] for more criteria for $\dot{W}^{1,p}$-imbedding domains, where they reduce the slice property to some weak slice properties.

It was noticed by Hajłasz [10] that the simple pointwise inequality

$$(1.1) \qquad\qquad |u(x) - u(y)| \le |x - y|^s [g(x) + g(y)]$$

can be used to characterize Sobolev functions u when $s = 1$. More generally, for $s \in (0, 1]$ and measurable function u, denote by $\mathcal{D}^s(u)$ the collection of all nonnegative measurable functions g such that (1.1) holds for all x, $y \in \Omega \setminus E$, where $E \subset \Omega$ satisfies $|E| = 0$. We also denote by $\mathcal{D}^s_{\mathrm{ball}}(u)$ the collection of all nonnegative measurable functions g such that (1.1) holds for all x, $y \in \Omega \setminus E$ satisfying $|x - y| < \frac{1}{2}\,\mathrm{dist}\,(x, \partial\Omega)$.

Definition 1.1. Let $s \in (0, 1]$ and $p \in (0, \infty]$.

(i) The homogeneous Hajłasz space $\dot{M}^{s,p}(\Omega)$ is the space of all measurable functions u such that

$$\|u\|_{\dot{M}^{s,p}(\Omega)} \equiv \inf_{g \in \mathcal{D}^s(u)} \|g\|_{L^p(\Omega)} < \infty.$$

(ii) The Sobolev-type Hajłasz space $\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$ is the space of all measurable functions u such that

$$\|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)} \equiv \inf_{g \in \mathcal{D}^s_{\mathrm{ball}}(u)} \|g\|_{L^p(\Omega)} < \infty.$$

Moreover, we set $M^{s,p}(\Omega) \equiv L^p(\Omega) \cap \dot{M}^{s,p}(\Omega)$ with $\|u\|_{M^{s,p}(\Omega)} \equiv \|u\|_{\dot{M}^{s,p}(\Omega)} + \|u\|_{L^p(\Omega)}$ for all $u \in M^{s,p}(\Omega)$, and similarly define $M^{s,p}_{\mathrm{ball}}(\Omega)$.

Obviously, for all $s \in (0, 1]$ and $p \in (0, \infty]$, $\dot{M}^{s,p}(\Omega) \subset \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$. Conversely, if Ω is a uniform domain, then $\dot{M}^{s,p}_{\mathrm{ball}}(\Omega) = \dot{M}^{s,p}(\Omega)$ for all $s \in (0, 1]$ and $p \in (n/(n+s), \infty]$; see [21, Theorem 19] and also [15, Lemma 14]. But, generally, we cannot expect that $\dot{M}^{s,p}(\Omega) = \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$. For example, this fails when $\Omega = B(0, 1) \setminus \{(x, 0) : x \ge 0\} \subset \mathbb{R}^2$.

Hajłasz-Sobolev spaces are closely related to the classical (Hardy-)Sobolev and Triebel-Lizorkin spaces. In fact, it was proved in [10, 21] that $\dot{W}^{1,p}(\Omega) = \dot{M}^{1,p}_{\mathrm{ball}}(\Omega)$ for $p \in (1, \infty]$ and $\dot{H}^{1,p}(\Omega) = \dot{M}^{1,p}_{\mathrm{ball}}(\Omega)$ for $p \in (n/(n+1), 1]$, which together with [33] implies that $\dot{M}^{1,p}(\mathbb{R}^n) = \dot{M}^{1,p}_{\mathrm{ball}}(\mathbb{R}^n) = \dot{F}^1_{p,2}(\mathbb{R}^n)$ for all $p \in (n/(n+1), \infty]$, while for all $s \in (0, 1)$ and $p \in (n/(n+s), \infty]$, $\dot{M}^{s,p}(\mathbb{R}^n) = \dot{M}^{s,p}_{\mathrm{ball}}(\mathbb{R}^n) = \dot{F}^s_{p,\infty}(\mathbb{R}^n)$ as proved in [37, 22, 23]. Here and in what follows, we always denote by $\dot{H}^{1,p}(\Omega)$ with $p \in (0, 1]$ the Hardy-Sobolev space as in [25], and by $\dot{F}^s_{p,q}(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and $p, q \in (0, \infty]$ the homogeneous Triebel-Lizorkin spaces as in [33].

Recently, it was proved in [14] (see [13, 28] and also Lemma 4.1 below) that for $p \in (1, \infty)$, Ω is a $W^{1,p}$-extension if and only if Ω is regular (see Definition 2.5) and $\dot{W}^{1,p}(\Omega) = \dot{M}^{1,p}(\Omega)$ (namely, $\dot{M}^{1,p}_{\mathrm{ball}}(\Omega) = \dot{M}^{1,p}(\Omega)$), while Ω is regular if and only if Ω is an $M^{1,p}$-extension domain. Some characterizations of the restriction of Besov and Triebel-Lizorkin spaces on regular domains were also established by Shvartsman [29]. Recall that it is an interesting subject to establish some intrinsic characterizations of $\dot{F}^s_{p,q}(\mathbb{R}^n)|_\Omega$, the restriction of the Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ on the domain Ω; see [26, 27, 33, 34]

for more discussions. In particular, some intrinsic characterizations of the restriction of Triebel-Lizorkin spaces on Lipschitz domains were established by Rychkov [26, 27] and Triebel [34].

In what follows, Ω is called an $\dot{M}_{\mathrm{ball}}^{s,p}$-imbedding domain if it supports an imbedding from $\dot{M}_{\mathrm{ball}}^{s,p}(\Omega)$ to $\dot{M}^{s-n/p,\infty}(\Omega)$ with $s \in (0, 1]$ and $p \in (n/s, \infty]$. We define $\dot{M}_{\mathrm{ball}}^{s,p}$-imbedding domains similarly.

The main purpose of this paper is to establish some geometric criteria for $\dot{M}_{\mathrm{ball}}^{s,p}$-extension (resp. $\dot{M}_{\mathrm{ball}}^{s,p}$-imbedding) domains of $\mathbb{R}^n$ with $n \geq 2$, $s \in (0, 1]$ and $p \in [n/s, \infty]$ (resp. $p \in (n/s, \infty]$). In particular, we prove that a bounded simply connected planar domain Ω is a weak α-cigar domain with $\alpha \in (0, 1)$ if and only if $\dot{F}_{p,\infty}^s(\mathbb{R}^2)|_\Omega = \dot{M}_{\mathrm{ball}}^{s,p}(\Omega)$ for some/all $s \in [\alpha, 1)$ and $p = (2-\alpha)/(s-\alpha)$.

More precisely, we first obtain the following conclusion by using some ideas from [5, 18, 16] and introducing a capacity associated to $\dot{M}_{\mathrm{ball}}^{s,n/s}(\Omega)$. See Section 3 for its proof.

Theorem 1.1. *If Ω is a bounded $\dot{M}_{\mathrm{ball}}^{s,n/s}$-extension domain for some $s \in (0, 1]$, then Ω has the LLC property.*

Recall that if a bounded simply connected planar domain, or a bounded domain of $\mathbb{R}^n$ with $n \geq 2$ that is quasiconformally equivalent to a uniform domain, has the LLC property, then it is a uniform domain; see [18]. We also recall that $\dot{F}_{p,\infty}^s(\mathbb{R}^n) = \dot{M}_{\mathrm{ball}}^{s,p}(\mathbb{R}^n)$ for all $s \in (0, 1)$ and $p \in (n/(n+s), \infty]$; see [37] and also [22]. Then as a corollary to Theorem 1.1, we have the following conclusion.

Corollary 1.1. *Let Ω be a bounded simply connected planar domain, or a bounded domain of $\mathbb{R}^n$ with $n \geq 2$ that is quasiconformally equivalent to a uniform domain. Then the following are equivalent:*

(i) Ω is a uniform domain;

(ii) Ω is an $\dot{M}_{\mathrm{ball}}^{s,n/s}$-extension domain for some/all $s \in (0, 1]$;

(iii) $\dot{F}_{n/s,\infty}^s(\mathbb{R}^n)|_\Omega = \dot{M}_{\mathrm{ball}}^{s,n/s}(\Omega)$ for some/all $s \in (0, 1)$.

When $p \in (n/s, \infty)$, we also establish the following geometric characterizations, which generalizes [2, Theorem 4.1] and [31, Theorem 1.1] to Hajłasz-Sobolev spaces. See Section 4 for its proof, which uses some ideas from [2, 19, 31], in particular, uses Theorems 4.1 and 4.2 below, and the weak self-improving property of a weak cigar domain established by Shvartsman in [31, Theorem 1.4] (see also Proposition 4.1 below).

Theorem 1.2. *(i) Let $\alpha \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ be a bounded weak α-cigar domain. Then for all $s \in (\alpha, 1]$ and $p \in [(n-\alpha)/(s-\alpha), \infty)$, Ω is an $\dot{M}_{\mathrm{ball}}^{s,p}$-extension domain and, especially, an $\dot{M}_{\mathrm{ball}}^{s,p}$-imbedding domain.*

(ii) Let $s \in (0, 1]$, $p \in (n/s, \infty)$ and $\alpha \in [(ps-n)/(p-1), 1]$. If Ω is an bounded $\dot{M}_{\mathrm{ball}}^{s,p}$-extension or $\dot{M}_{\mathrm{ball}}^{s,p}$-imbedding domain having the slice property, then Ω is a weak α-cigar domain.

At the endpoint case $p = \infty$, as proved by Gehring and Martio [5], a bounded domain Ω is a weak α-cigar domain with $\alpha \in (0, 1]$ if and only if it is an $\dot{M}_{\mathrm{ball}}^{\alpha,\infty}$-extension domain,

and if and only if it is an $\dot{M}_{\mathrm{ball}}^{\alpha,\infty}$-imbedding domain, where $\dot{M}^{\alpha,\infty}(\Omega)$ and $\dot{M}_{\mathrm{ball}}^{\alpha,\infty}(\Omega)$ coincide with $\mathrm{Lip}_\alpha(\Omega)$ and $\mathrm{loc\,Lip}_\alpha(\Omega)$ as in [5], respectively. Recall that a bounded simply connected planar domain, or a bounded domain of $\mathbb{R}^n$ with $n \geq 2$ that is quasiconformally equivalent to a uniform domain, always has the slice property (see [2]). Then, as a corollary to Theorem 1.2 and [5], we have the following conclusion, which together with Corollary 1.1 gives an intrinsic characterization of the restriction of the Triebel-Lizorkin space $\dot{F}_{p,\infty}^s(\mathbb{R}^n)|_\Omega$ for a class of domains Ω.

Corollary 1.2. *Let $\alpha \in (0,\,1)$ and Ω be a bounded simply connected planar domain, or a bounded domain of $\mathbb{R}^n$ with $n \geq 2$ that is quasiconformally equivalent to a uniform domain. Then the following are equivalent:*

(i) Ω is a weak α-cigar domain;

(ii) $\dot{F}_{p,\infty}^s(\mathbb{R}^n)|_\Omega = \dot{M}_{\mathrm{ball}}^{s,p}(\Omega)$ for some/all $s \in [\alpha,\,1)$ and $p = (n-\alpha)/(s-\alpha)$;

(iii) Ω is an $\dot{M}_{\mathrm{ball}}^{s,p}$-extension domain for some/all $s \in [\alpha,\,1]$ and $p = (n-\alpha)/(s-\alpha)$;

(iv) Ω is an $\dot{M}_{\mathrm{ball}}^{s,p}$-imbedding domain for some/all $s \in [\alpha,\,1]$ and $p = (n-\alpha)/(s-\alpha)$.

Finally, let $\alpha \in (0,\,1)$ and Ω be a bounded weak α-cigar domain, namely, bounded α-subhyperbolic domain as in [31]. Then, with the aid of its weak self-improving property established in [31, Theorem 1.5] (see also Proposition 4.1 below), Shvartsman [31, Theorem 1.1] proved that Ω is a $W^{1,p}$-extension domain for every $p \in ((n-\alpha^*)/(1-\alpha^*),\,\infty)$ with $\alpha^* \in (0,\,\alpha)$ as in Proposition 4.1. Following this and [31, Theorems 1.1] with taking $p \in ((n-\alpha^*)/(1-\alpha^*),\,(n-\alpha)/(1-\alpha))$, if Ω is also a finitely connected planar domain, then Shvartsman [31, p. 2210] pointed out that Ω is a weak τ-cigar domain with $\tau \in (\alpha^*,\,\alpha)$, while such a self-improving property for an arbitrary weak cigar domain is still unknown. However, as Shvartsman informed me, when $p \in ((n-\alpha^*)/(1-\alpha^*),\,(n-\alpha)/(1-\alpha))$, the proof of [31, Theorem 1.1] actually proved a weak form of [31, Theorem 1.1]: there exists a continuous linear extension operator from $W^{1,p}(\Omega) \cap W^{1,\,(n-\alpha)/(1-\alpha)}(\Omega)$ to $W^{1,p}(\Omega)$. But, the following conclusion shows that Ω does have the self-improving property when it is also a finitely connected planar domain. The point is that, as observed in Remark 4.3, if Ω is a finitely connected planar domain or more generally, Ω has the slice property, then the $\dot{M}_{\mathrm{ball}}^{s,p}(\Omega)$-imbedding required in Theorem 1.2(ii) can be reduced to a weaker one, which is already obtained in the proof of Theorem 1.2(i) with the aid of Shvartsman [31, Theorem 1.5] or which, when $s = 1$, can also be deduced from the above weak form of [31, Theorem 1.1] as Shvartsman informed me. See Section 4 for more details.

Theorem 1.3. *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a finitely connected bounded planar domain or a bounded domain that is quasiconformally equivalent to a uniform domain, or more generally, be a bounded domain satisfying the slice property. If Ω is a weak α-cigar domain with some $\alpha \in (0,\,1)$, then Ω is a weak α^*-cigar domain with $\alpha^* \in (0,\,\alpha)$, and hence for all $s \in (\alpha^*,\,1]$ and $p \in [(n-\alpha^*)/(s-\alpha^*),\,\infty)$, Ω is an $\dot{M}_{\mathrm{ball}}^{s,p}$-extension domain and, especially, an $\dot{M}_{\mathrm{ball}}^{s,p}$-imbedding domain.*

2   Preliminaries

In this section, we recall some notions and basic properties of domains and Hajłasz spaces. We begin with the notion of a uniform domain.

Definition 2.1. A domain $\Omega \subset \mathbb{R}^n$ is called a uniform domain if there exists a positive constant C such that for all $x, y \in \Omega$, there exists a rectifiable curve $\gamma : [0, T] \to \Omega$, parameterized by the arc length, with $\gamma(0) = x$ and $\gamma(T) = y$, and satisfying that $T \leq C|x - y|$ and

$$(2.1) \qquad \bigcup_{t \in [0, T]} B\left(\gamma(t), \frac{1}{C} \min\{t, T - t\}\right) \subset \Omega.$$

Closely related to the concept of a uniform domain, Gehring [4] introduced the notion of linear local connectivity.

Definition 2.2. A domain $\Omega \subset \mathbb{R}^n$ is said to have the linearly locally connectivity (for short, LLC) property if there exists a constant $b \in (0, 1]$ such that for all $z \in \mathbb{R}^n$ and $r > 0$,
LLC(1)   points in $\Omega \cap B(z, r)$ can be joined in $\Omega \cap B(z, r/b)$;
LLC(2)   points in $\Omega \setminus B(z, r)$ can be joined in $\Omega \setminus B(z, br)$.

It is known that each uniform domain has the LLC property. Conversely, assume that Ω is a bounded simply connected planar domain, or a bounded domain of $\mathbb{R}^n$ with $n \geq 3$ that is quasiconformally equivalent to a uniform domain. If Ω has the LLC property, then it is a uniform domain; see [35] and also [18].

Now we recall the notion of a weak cigar domain; see [5, 2].

Definition 2.3. Let $\alpha \in (0, 1]$. Then a domain $\Omega \subset \mathbb{R}^n$ is called a weak α-cigar domain if there exists a positive constant C such that for every pair of points $x, y \in \Omega$, there exists a rectifiable curve $\gamma \subset \Omega$ joining x and y, and satisfying

$$\int_\gamma [d(z, \Omega^\complement)]^{\alpha - 1} |dz| \leq C|x - y|^\alpha.$$

Notice that the class of weak 1-cigar domains coincides with the class of quasiconvex domains, and a bounded weak α-cigar domain is a weak β-cigar domain for all $\beta \in (\alpha, 1]$; see [2] for details. Moreover, a uniform domain is also a weak α-cigar domain for all $\alpha \in (0, 1]$. .

The following slice property was introduced by Buckley and Koskela [2]. In what follows, for every rectifiable curve γ, we denote its length by $\ell(\gamma)$.

Definition 2.4. A domain Ω has a slice property with respect to $C > 1$ if for every pair of points $x, y \in \Omega$, there exists a rectifiable curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$, and pairwise disjoint collection of open subsets $\{S_i\}_{i=0}^j$, $j \geq 0$, of Ω such that
   (i) $x \in S_0$, $y \in S_j$ and x and y are in different components of $\Omega \setminus \overline{S_i}$ for $0 < i < j$;
   (ii) if $F \subset\subset \Omega$ is a curve containing both x and y, and $0 < i < j$, then $\operatorname{diam}(S_i) \leq C\ell(F \cap S_i)$;
   (iii) for $0 \leq t \leq 1$, $B(\gamma(t), C^{-1}d(\gamma(t), \Omega^\complement)) \subset \cup_{i=0}^j S_i$;
   (iv) if $0 \leq i \leq j$, then $\operatorname{diam} S_i \leq Cd(z, \Omega^\complement)$ for all $z \in \gamma_i \equiv \gamma \cap S_i$; also, there exists $x_i \in S_i$ such that $x_0 = x$, $x_j = y$ and $B(x_i, C^{-1}d(x_i, \Omega^\complement)) \subset S_i$.

It was proved by Buckley and Koskela [2] that every simply connected domain in $\mathbb{R}^2$, or every domain in $\mathbb{R}^n$ with $n \geq 3$ that is quasiconformally equivalent to a uniform domain, has the slice property as in Definition 2.4.

We also recall the notion of the regularity of a domain.

Definition 2.5. *A domain $\Omega \subset \mathbb{R}^n$ is regular if there exist positive constants θ and C such that for all $x \in \overline{\Omega}$ and $r \in (0, \theta)$, $|B(x, r) \cap \Omega| \geq C|B(x, r)|$.*

We point out that the regularity of Ω does not depend on the choice of θ and C in the following sense: if Ω is regular with θ and C, then for any $\theta' \in (0, \infty)$, there exists a constant C' such that Ω is regular with θ' and C'.

The following lemma established in [21] will be useful in the following proofs. In what follows, for every $\rho \in (0, \infty]$, similarly to $\mathcal{D}^s_{\mathrm{ball}}(u)$, we denote by $\mathcal{D}^{s,\rho}_{\mathrm{ball}}(u)$ the collection of all measurable functions g such that (1.1) holds for all $x, y \in \Omega \setminus E$ satisfying $|x - y| < \rho \operatorname{dist}(x, \partial\Omega)$. Notice that $\mathcal{D}^s_{\mathrm{ball}}(u) = \mathcal{D}^{s,1/2}_{\mathrm{ball}}(u)$ and $\mathcal{D}^s(u) = \mathcal{D}^{s,\infty}_{\mathrm{ball}}(u)$.

Lemma 2.1. *Let $s \in (0, 1]$ and $p \in (n/(n + s), \infty]$. Then $u \in \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$ if and only if there exists a $\rho \in (0, 1)$ such that $\inf_{g \in \mathcal{D}^{s,\rho}_{\mathrm{ball}}(u)} \|g\|_{L^p(\Omega)} < \infty$. Moreover, for given ρ, there exists a positive constant C such that for all $u \in \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$,*

$$C^{-1}\|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)} \leq \inf_{g \in \mathcal{D}^{s,\rho}_{\mathrm{ball}}(u)} \|g\|_{L^p(\Omega)} \leq C\|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)}.$$

Finally, we state some conventions. Throughout the paper, we denote by C a positive constant which is independent of the main parameters, but which may vary from line to line. Constants with subscripts, such as C_0, do not change in different occurrences. The symbol $A \lesssim B$ or $B \gtrsim A$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$. For any locally integrable function f, we denote by $\fint_E f \, d\mu$ the average of f on E, namely, $\fint_E f \, d\mu \equiv \frac{1}{|E|} \int_E f \, dx$.

3   Proof of Theorem 1.1

To prove Theorem 1.1, we need the following capacity.

Definition 3.1. *Let $s \in (0, 1]$. For every pair $E, F \subset \Omega$ of disjoint continua, define the capacity associated to $\dot{M}^{s,n/s}_{\mathrm{ball}}(\Omega)$ by*

$$\operatorname{Cap}_{\dot{M}^{s,n/s}_{\mathrm{ball}}}(E, F, \Omega) \equiv \inf_{u \in \Delta_s(E,F,\Omega)} \|u\|^{n/s}_{\dot{M}^{s,n/s}_{\mathrm{ball}}(\Omega)},$$

where $\Delta_s(E, F, \Omega)$ denotes the collection of all continuous functions $u \in \dot{M}^{s,n/s}_{\mathrm{ball}}(\Omega)$ with $u(x) = 0$ for all $x \in E$ and $u(x) = 1$ for all $x \in F$.

Obviously, for every pair $E, F \subset \Omega$ of disjoint continua and every pair $\widetilde{E}, \widetilde{F} \subset \mathbb{R}^n$ of disjoint continua satisfying $E \subset \widetilde{E}$ and $F \subset \widetilde{F}$, we have

(3.1) $\operatorname{Cap}_{\dot{M}^{s,n/s}_{\mathrm{ball}}}(E, F, \Omega) \leq \operatorname{Cap}_{\dot{M}^{s,n/s}_{\mathrm{ball}}}(\widetilde{E}, \widetilde{F}, \mathbb{R}^n).$

Moreover, a reverse inequality also follows for $\dot{M}^{s,n/s}_{\mathrm{ball}}$-extension domains by modifying the proof of [18, Theorem 2.2]. We omit the details.

Lemma 3.1. *If Ω is an $\dot{M}_{\mathrm{ball}}^{s,\,n/s}$-extension domain, then there exists a positive constant C such that for every pair $E,\,F \subset \Omega$ of disjoint continua,*

$$\mathrm{Cap}_{\dot{M}_{\mathrm{ball}}^{s,\,n/s}}(E,\,F,\,\mathbb{R}^n) \leq C\,\mathrm{Cap}_{\dot{M}_{\mathrm{ball}}^{s,\,n/s}}(E,\,F,\,\Omega).$$

The following property of the capacity plays an important role in the proof of Theorem 1.1 and it is proved by using some ideas of [16, Theorem 5.9].

Lemma 3.2. *Let $s \in (0,\,1]$ and $\delta \in (0,\,\infty)$. There exists a positive constant C such that for every pair $E,\,F \subset \mathbb{R}^n$ of disjoint continua, if $\min\{\,\mathrm{diam}\,E,\,\mathrm{diam}\,F\} \geq \delta\,\mathrm{dist}\,(E,\,F)$, then $\mathrm{Cap}_{\dot{M}_{\mathrm{ball}}^{s,\,n/s}}(E,\,F,\,\mathbb{R}^n) \geq C$.*

Proof. Notice that if continua $\widetilde{F} \subset F$ and $\widetilde{E} \subset E$, then $\Delta_s(E,\,F,\,\mathbb{R}^n) \subset \Delta_s(\widetilde{E},\,\widetilde{F},\,\mathbb{R}^n)$ and thus

$$\mathrm{Cap}_{\dot{M}_{\mathrm{ball}}^{s,\,n/s}}(E,\,F,\,\mathbb{R}^n) \geq \mathrm{Cap}_{\dot{M}_{\mathrm{ball}}^{s,\,n/s}}(\widetilde{E},\,\widetilde{F},\,\mathbb{R}^n).$$

So without loss of generality, we may assume that $\mathrm{diam}\,E = \mathrm{diam}\,F \geq \delta\,\mathrm{dist}\,(E,\,F)$.

Fix $x_0 \in F$ and $r \equiv (2 + \delta)\,\mathrm{diam}\,E$. Then $E,\,F \subset B(x_0,\,r)$. Let $u \in \Delta_s(E,\,F,\,\mathbb{R}^n)$ and, without loss of generality, assume that $u_{B(x_0,\,r)} \leq 1/2$. Then for every $x \in F$ and $g \in \mathcal{D}_{\mathrm{ball}}^s(u) \cap L^{n/s}(\Omega)$, we have

$$\begin{aligned}
\frac{1}{2} \leq |u(x) - u_{B(x_0,\,r)}| &\leq \sum_{i=-1}^{\infty} |u_{B(x,\,2^{-i}r)} - u_{B(x,\,2^{-i-1}r)}| + |u_{B(x,\,2r)} - u_{B(x_0,\,r)}| \\
&\leq \sum_{i=-1}^{\infty} (2^{-i}r)^s \left(\fint_{B(x,\,2^{-i}r)} [g(z)]^{n/s}\,dz \right)^{s/n} \\
&\leq \sum_{i=-1}^{\infty} (2^{-i}r)^{s/n} \left(\frac{1}{2^{-i}r} \int_{B(x,\,2^{-i}r)} [g(z)]^{n/s}\,dz \right)^{s/n} \\
&\lesssim \sup_{0 < t \leq 2r} \left(\frac{r}{t} \int_{B(x,\,t)} [g(z)]^{n/s}\,dz \right)^{s/n},
\end{aligned}$$

which implies that there exists $t_x \in (0,\,2r]$ such that

$$t_x \lesssim r \int_{B(x,\,t)} [g(z)]^{n/s}\,dz.$$

By the Vitali covering lemma, we can find points $\{x_i\}_i \subset F$ such that $\{B(x_i,\,t_{x_i})\}_i$ are pairwise disjoint and $F \subset \cup_i 5B(x_i,\,t_{x_i})$. Thus,

$$\mathrm{diam}\,F \leq \sum_i 10 t_{x_i} \lesssim r \sum_i \int_{B(x_i,\,t_{x_i})} [g(z)]^{n/s}\,dz \lesssim r \int_{B(x_0,\,3r)} [g(z)]^{n/s}\,dz,$$

which yields that $\|u\|_{\dot{M}_{\mathrm{ball}}^{s,\,p}(\Omega)} \gtrsim 1$ and thus finishes the proof of Lemma 3.2. $\qquad\square$

Proof of Theorem 1.1. We first prove that Ω has the LLC(2) property. Let x_1, $x_2 \in B(x_0, r) \cap \Omega$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$. Suppose that x_1 and x_2 are not in the same component of $\Omega \setminus B(x_0, b_0 r)$ with $b_0 \in (0, 1/4)$. It then suffices to prove that b_0 is bounded from below. To this send, we choose a rectifiable curve $\gamma \subset \Omega$ joining x_1 and x_2, and denote by F_i the component of $\gamma \cap (\Omega \setminus B(x_0, r/2))$ containing x_i for $i = 1$, 2. Obviously, $\operatorname{diam} F_i \geq r/2 \geq \operatorname{dist}(F_1, F_2)/4$ for $i = 1$, 2. Then by (3.1) and Lemmas 3.1 and 3.2, we have

$$(3.2) \qquad \operatorname{Cap}_{\dot{M}^{s,\,n/s}_{\mathrm{ball}}}(F_1, F_2, \Omega) \sim \operatorname{Cap}_{\dot{M}^{s,\,n/s}_{\mathrm{ball}}}(F_1, F_2, \mathbb{R}^n) \gtrsim 1.$$

To estimate $\operatorname{Cap}_{\dot{M}^{s,\,n/s}_{\mathrm{ball}}}(F_1, F_2, \Omega)$ from above, for all $x \in \mathbb{R}^n$, define

$$u(x) \equiv \begin{cases} 1, & x \in \Omega \cap B(x_0, b_0 r); \\ \left(\log \frac{1}{2b_0}\right)^{-1} \left(\log \frac{r}{2|x-x_0|}\right), & x \in \Omega \cap (B(x_0, r/2) \setminus B(x_0, b_0 r)); \\ 0, & x \in \Omega \setminus B(x_0, r/2) \end{cases}$$

and

$$g(x) \equiv \frac{1}{|x - x_0|^s} \left(\log \frac{1}{2b_0}\right)^{-1} \chi_{\overline{\Omega \cap \overline{B(x_0, r/2)} \setminus B(x_0, b_0 r)}}(x).$$

Then we claim that there exists a positive constant independent of u, x_0, b_0, r such that Cg is an element of $\mathcal{D}^{s,\,1/32}_{\mathrm{ball}}(u)$. Assume that the claim holds for the moment. Then by Lemma 2.1, $u \in \dot{M}^{s,\,p}_{\mathrm{ball}}(\Omega)$ and

$$\|u\|_{\dot{M}^{s,\,n/s}_{\mathrm{ball}}(\Omega)} \leq \|g\|_{L^{n/s}(\Omega)}$$

$$\lesssim \left(\log \frac{1}{2b_0}\right)^{-1} \left\{ \int_{B(x_0,\,r/2) \setminus B(x_0,\,b_0 r)} |z - x_0|^{-n} \, dz \right\}^{s/n}$$

$$\lesssim \left(\log \frac{1}{2b_0}\right)^{s/n - 1}.$$

Moreover, observe that $u \in \Delta_s(F_1, F_2, \Omega)$. So we have

$$(3.3) \qquad \operatorname{Cap}_{\dot{M}^{s,\,n/s}_{\mathrm{ball}}}(E, F, \Omega) \lesssim \left(\log \frac{1}{2b_0}\right)^{1 - n/s},$$

which together with (3.2) implies that $b_0 \gtrsim 1$ and hence reduces the LLC(2) property of Ω to proving the above claim.

To prove the above claim, it suffices to prove that for all $x, y \in \Omega$ satisfying $|x - y| \leq \min\{d(x, \Omega^{\complement}), d(x, \Omega^{\complement})\}/16$,

$$|u(x) - u(y)| \lesssim |x - y|^s [g(x) + g(y)].$$

Fix such $x, y \in \Omega$. Without loss of generality, we assume that $0 \leq u(y) < u(x) \leq 1$. Then $x \in \Omega \cap \overline{B(x_0, r/2)}$ and by $\overline{B(x_0, b_0 r)} \cap \Omega^{\complement} \neq \emptyset$, we have $d(x, \Omega^{\complement}) \leq |x - x_0| + b_0 r < 3r/4$.

It will not happen that $x \in \Omega \cap B(x_0, b_0 r)$ and $y \in \Omega \setminus B(x_0, r/2)$ since, in this case, $|x - y| \geq r/4 \geq d(x, \Omega^\complement)/3$. If $x, y \in \Omega \cap \overline{B(x_0, r/2)} \setminus B(x_0, b_0 r)$, then

$$|u(x) - u(y)| = \left(\log \frac{1}{2b_0} \right)^{-1} \left(\log \frac{|y - x_0|}{|x - x_0|} \right) \lesssim \left(\log \frac{r}{2b_0} \right)^{-1} \frac{|x - y|^s}{|x - x_0|^s}.$$

If $x \in \Omega \cap \overline{B(x_0, b_0 r)}$ and $y \in \Omega \cap \overline{B(x_0, r/2)} \setminus B(x_0, b_0 r)$,

$$|u(x) - u(y)| = \left| 1 - \left(\log \frac{r}{2b_0} \right)^{-1} \left(\log \frac{r}{2|y - x_0|} \right) \right|$$

$$\lesssim \left(\log \frac{1}{2b_0} \right)^{-1} \frac{|y - x_0| - b_0 r}{|y - x_0|} \lesssim \left(\log \frac{r}{2b_0} \right)^{-1} \frac{|x - y|^s}{|y - x_0|^s}.$$

If $x \in \Omega \cap \overline{B(x_0, r/2)} \setminus B(x_0, b_0 r)$ and $y \in \Omega \setminus B(x_0, r/2)$,

$$|u(x) - u(y)| = \left(\log \frac{r}{2b_0} \right)^{-1} \left(\log \frac{r}{2|x - x_0|} \right)$$

$$\lesssim \left(\log \frac{1}{2b_0} \right)^{-1} \frac{r/2 - |x - x_0|}{|x - x_0|} \lesssim \left(\log \frac{r}{2b_0} \right)^{-1} \frac{|x - y|^s}{|x - x_0|^s}.$$

This shows the above claim and thus proves that Ω has the LLC(2) property.

To prove that Ω has LLC(1) property, it suffices to prove that Ω is quasiconvex, namely, for every pair x_1, x_2 of points in Ω, there exists a curve $\gamma \subset \Omega$ joining them with $\ell(\gamma) \leq C|x - y|$, where the constant C is independent of x, y and γ.

To this end, let x_1, x_2 be a pair of points in Ω. If $|x_1 - x_2| < \max\{d(x_1, \Omega^\complement), d(x_2, \Omega^\complement)\}$, then the line segment joining x_1 and x_2 is the desired curve. Assume that $|x_1 - x_2| \geq \max\{d(x_1, \Omega^\complement), d(x_2, \Omega^\complement)\}$. Let $\gamma^{(0)} \subset \Omega$ be a curve joining x_1 and x_2, and let F_i be the component containing x_i of $\gamma^{(0)} \cap B(x_i, |x_1 - x_2|/4)$ for $i = 1, 2$. Notice that (3.2) still holds by the same argument. Moreover, there exists a positive constant $N_0 > 1$ independent of $x_1, x_2, \gamma^{(0)}, F_1, F_2$ such that F_1, F_2 are in the same component of $\Omega \cap \overline{B(x_1, N_0|x_1 - x_2|)}$. To see this, assume that F_1, F_2 are not in the same component of $\Omega \cap \overline{B(x_1, N|x_1 - x_2|)}$ for some $N > 2$. Then $\overline{B(x_1, N|x_1 - x_2|)} \cap \Omega^\complement \neq \emptyset$ and hence by an argument similar to the proof of (3.3), we have

$$\mathrm{Cap}_{\dot{M}^{s, n/s}_{\mathrm{ball}}}(E, F, \Omega) \lesssim (\log N)^{1 - n/s},$$

which means $N \lesssim 1$ and hence shows the existence of N_0. Therefore, letting L be the infimum of the length of all curves joining F_1 and F_2, we have $L < \infty$ and then define the function

$$v(x) \equiv L^{-1} \inf_\gamma \ell(\gamma \cap B(x_1, N_0|x_1 - x_2|))$$

for all $x \in \Omega$, where the infimum is taken over all the rectifiable curves $\gamma \subset \Omega$ joining x and F_1. For all $x \in \Omega$, define

$$h(x) \equiv C L^{-1} (N_0|x_1 - x_2|)^{1 - s} \chi_{\Omega \cap \overline{B(x_1, N_0|x_1 - x_2|)}}(x).$$

Then we claim that there exists a positive constant C independent of v, L, x_1, x_2, N_0 such that Ch is an element of $\mathcal{D}_{\text{ball}}^{s,\,1/16}(v)$. Assume that this claim holds for the moment. Set $\widetilde{v} \equiv \min\{v,\, 1\}$. Then h is also a constant multiple of an element of $\mathcal{D}_{\text{ball}}^{s,\,1/16}(\widetilde{v})$, which together with Lemma 2.1 implies that $\widetilde{v} \in \dot{M}_{\text{ball}}^{s,\,n/s}(\Omega)$ and $\|\widetilde{v}\|_{\dot{M}_{\text{ball}}^{s,\,n/s}(\Omega)} \lesssim \|h\|_{L^{n/s}(\Omega)} \lesssim$ $(N_0|x_1 - x_2|/L)^{n/s}$. Since $\widetilde{v} \in \Delta_s(F_1,\, F_2,\, \Omega)$, we then have

$$1 \lesssim \mathrm{Cap}_{\dot{M}_{\text{ball}}^{s,\,n/s}}(F_1,\, F_2,\, \Omega) \lesssim \left(\frac{N_0 r}{L} \right)^{n/s},$$

and thus $L \lesssim N_0|x_1 - x_2|$. So we can find a rectifiable curve $\gamma^{(1)}$ joining x_1, x_2.

Now we prove the claim that h is a constant multiple of an element of $\mathcal{D}_{\text{ball}}^{s,\,1/16}(v)$. To this end, we only need to chace that for x, $y \in \Omega$ with $|x - y| \le d(x,\, \Omega^{\complement})/16$,

(3.4) $$|u(x) - u(y)| \lesssim |x - y|^s [h(x) + h(y)].$$

If x, y lie in the same component of $\Omega \setminus \overline{B(x_1,\, N_0|x_1 - x_2|)}$, then $u(x) = u(y)$ and thus (3.4) holds. Assume that x, y lie in the different components of $\Omega \setminus \overline{B(x_1,\, N_0|x_1 - x_2|)}$. Then the line segment joining x and y has a nonempty intersection with $B(x_1,\, N_0|x_1 - x_2|)$ and assume it contains w. Moreover, $|u(x) - u(y)| \le |x - y|/L$. Since

$$d(x,\, \Omega^{\complement}) \le |x - w| + d(w,\, \Omega^{\complement}) \le |x - y| + d(w,\, \Omega^{\complement}) \le d(x,\, \Omega^{\complement})/16 + d(w,\, \Omega^{\complement})$$

and $d(w,\, \Omega^{\complement}) \le 2N_0|x_1 - x_2|$ imply that $|x - y| \le d(x,\, \Omega^{\complement}) \lesssim d(w,\, \Omega^{\complement}) \lesssim N_0|x_1 - x_2|$, so (3.4) holds and thus gives the above claim.

Moreover, without loss of generality, we may assume that $\gamma^{(1)} \cap F_i$ consists of a unique point, $x_i^{(1)}$, for $i = 1,\, 2$. Let $F_i^{(1)}$ be the component of $\gamma^{(1)} \cap B(x_i,\, |x_i - x_i^{(1)}|/4)$ containing x_i, and E_i be the component of $\gamma^{(1)} \cap B(x_i^{(1)},\, |x_i - x_i^{(1)}|/4)$ containing $x_i^{(1)}$. Then repeating the above procedure we can find a curve $\gamma_i^{(2)}$ joining $F_i^{(1)}$ and E_i such that $\ell(\gamma_i^{(2)}) \lesssim$ $|x_i - x_i^{(1)}|$. Denote by $x_i^{(2)}$ the unique point of $\gamma_i^{(2)} \cap F_i$. Then $|x_i - x_i^{(2)}| \le |x_i - x_i^{(1)}|/4 \le$ $|x_1 - x_2|/4^2$, $\gamma^{(1)} \cup \gamma_1^{(2)} \cup \gamma_2^{(2)}$ contains a curve $\gamma^{(2)}$ joining $x_1^{(2)}$ and $x_2^{(2)}$ with

$$\ell(\gamma^{(2)}) \lesssim |x_1 - x_2| + \sum_{i=1,\,2} |x_i - x_i^{(1)}|/2 \lesssim [1 + 1/2]|x_1 - x_2|.$$

Repeating this procedure k times until $|x_i - x_i^{(k)}| < d(x_i,\, \Omega^{\complement})$ for $i = 1,\, 2$, we obtain a curve $\gamma^{(k)} \subset \Omega$ joining $x_1^{(k)}$ and $x_2^{(k)}$ with

$$\ell(\gamma^{(k)}) \lesssim |x_1 - x_2| + \sum_{j=1}^{k} \sum_{i=1,\,2} |x_i - x_i^{(j)}|/2 \lesssim |x_1 - x_2| \sum_{j=0}^{k} (1 + 2 \times 4^{-j}) \lesssim |x_1 - x_2|.$$

Let γ be the union of $\gamma^{(k)}$, the line segment joining x_1 and $x_1^{(k)}$ and the line segment joining x_2 and $x_2^{(k)}$. Then we know $\gamma \subset \Omega$ joins x_1 and x_2, and $\ell(\gamma) \lesssim |x_1 - x_2|$, which is as desired and thus completes the proof of Theorem 1.1. $\qquad\square$

4 Proofs of Theorems 1.2 and 1.3

The proofs of Theorems 1.2 and 1.3 consist of a sequence of auxiliary conclusions, in particular, Theorem 4.1, Theorem 4.2 and Proposition 4.1 below.

We begin with several equivalent characterizations of Hajłasz-Sobolev imbeddings, whose proof borrows some ideas from [20, 19, 11, 14]. In what follows, for $R \in (0, \infty)$ and $u \in L^1_{\mathrm{loc}}(\Omega)$, we define the maximal function $\mathcal{M}^\Omega_R(u)(x)$ for all $x \in \Omega$ by

$$\mathcal{M}^\Omega_R(u)(x) \equiv \sup_{r \in (0, R)} \frac{1}{|B(x, r) \cap \Omega|} \int_{B(x, r) \cap \Omega} |u(z)| \, dz.$$

Theorem 4.1. *Let $s \in (0, 1]$ and $p \in (n/s, \infty)$. Then the following are equivalent:*

(i) Ω is an $M^{s,p}_{\mathrm{ball}}$-imbedding domain;

(ii) Ω supports the imbedding from $M^{s,p}_{\mathrm{ball}}(\Omega)$ to $\dot{M}^{s-n/p,\infty}(\Omega)$;

(iii) There exist positive constants δ_1 and C such that for all $u \in M^{s,p}_{\mathrm{ball}}(\Omega)$ and almost all $x, y \in \Omega$ with $|x - y| \leq \delta_1$,

$$|u(x) - u(y)| \leq C|x - y|^{s-n/p}\|u\|_{M^{s,p}_{\mathrm{ball}}(\Omega)};$$

(iv) There exist positive constants δ_2 and C such that for all $u \in \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$ and almost all $x, y \in \Omega$ with $|x - y| \leq \delta_2$,

$$|u(x) - u(y)| \leq C|x - y|^{s-n/p}\|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)}; \tag{4.1}$$

(v) There exist positive constants δ_3, N_1 and C such that for all $u \in \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$, $g \in \mathcal{D}^s_{\mathrm{ball}}(u)$ and almost all $x, y \in \Omega$ with $|x - y| \leq \delta_3$,

$$|u(x) - u(y)| \leq C|x - y|^{s-n/p}\left\{\int_{B(x, N_1|x-y|) \cap \Omega} [g(z)]^p \, dz\right\}^{1/p}; \tag{4.2}$$

(vi) Ω is regular and there exist positive constants δ_4, N_2 and C such that for all $u \in \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$, $g \in \mathcal{D}^s_{\mathrm{ball}}(u)$ and almost all $x, y \in \Omega$ with $|x - y| \leq \delta_4$,

$$|u(x) - u(y)| \leq C|x - y|^s \left\{\mathcal{M}^\Omega_{N_2|x-y|}(g^p)(x) + \mathcal{M}^\Omega_{N_2|x-y|}(g^p)(y)\right\}^{1/p}. \tag{4.3}$$

Moreover, if Ω is bounded, then it is an $\dot{M}^{s,p}_{\mathrm{ball}}$-imbedding domain if and only if one/all of (i) through (vi) holds.

Proof of Theorem 4.1. We first notice that if Ω is bounded, then (iv) means that Ω is an $\dot{M}^{s,p}_{\mathrm{ball}}$-imbedding domain. So it suffices to prove the equivalence of (i) through (vi). Obviously, (i)$\Rightarrow$(ii)$\Rightarrow$(iii). In what follows, we will prove that (iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(vi)$\Rightarrow$(iv)$\Rightarrow$(i).

(iii)$\Rightarrow$(iv). Let $u \in \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$ and $g \in \mathcal{D}^s_{\mathrm{ball}}(u)$ with $\|g\|_{L^p(\Omega)} \lesssim \|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)}$. Let $x, y \in \Omega$ be any pair of points satisfying $|x - y| < \delta_1$. Without loss of generality, we may assume that $u(y) < u(x)$ and $u(y) \leq u(z) \leq u(x)$ for all $z \in \Omega$. In fact, set

$$v(z) = \begin{cases} u(x) & \text{if } u(z) > u(x); \\ u(z) & \text{if } u(y) \leq u(z) \leq u(x); \\ u(y) & \text{if } u(z) < u(y). \end{cases}$$

Then $\mathcal{D}^s_{\mathrm{ball}}(u) \subset \mathcal{D}^s_{\mathrm{ball}}(v)$ and thus $\|v\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)} \lesssim \|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)}$. So it suffices to prove (4.1) for v. Moreover, we may assume that $u(y) \geq 0$. In fact, if $u(x) \leq 0$, then we only need to consider $-u$. If $u(y) < 0 \leq u(x)$, then let $u_1 = u\chi_{\{z\in\Omega:\, u(z)\geq 0\}}$ and $u_2 = -u\chi_{\{z\in\Omega:\, u(z)\leq 0\}}$. Notice that $\mathcal{D}^s_{\mathrm{ball}}(u) \subset \mathcal{D}^s_{\mathrm{ball}}(u_1) \cap \mathcal{D}^s_{\mathrm{ball}}(u_2)$, which implies that $\|u_1\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)} + \|u_2\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)} \lesssim \|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)}$, and $|u(x) - u(y)| \leq |u_1(x) - u_1(y)| + |u_2(x) - u_2(y)|$. So we only need to prove (4.1) for u_1 and u_2.

Let φ be a smooth function satisfying that $\varphi(z) = 1$ for $z \in B(x, \delta_1)$, $\operatorname{supp}\varphi \subset B(x, 10\delta_1)$, $0 \leq \varphi(z) \leq 1$ and $|\nabla\varphi(z)| \leq 100/\delta_1$ for all $z \in \mathbb{R}^n$. Define $v(z) \equiv [u(z) - u(y)]\varphi(z)$ for all $z \in \Omega$. Then it is easy to check that $v \in M^{s,p}_{\mathrm{ball}}(\Omega)$ and

$$g\varphi + 100(\delta_1)^{-s}[u(x) - u(y)]\chi_{\{B(x,\,10\delta_1)\cap\Omega\}} \in \mathcal{D}^s_{\mathrm{ball}}(v),$$

which together with $u(y) \leq u(z) \leq u(x)$ for all $z \in \Omega$ implies that

$$\|v\|_{M^{s,p}_{\mathrm{ball}}(\Omega)} \lesssim \|g\varphi\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \lesssim \|u\|_{M^{s,p}_{\mathrm{ball}}(\Omega)} + [(\delta_1)^{-s+n/p} + (\delta_1)^{n/p}][u(x) - u(y)].$$

Thus, by (iii),

$$\begin{aligned}
|u(x) - u(y)| &= |v(x) - v(y)| \\
&\lesssim |x - y|^{s-n/p}\|v\|_{M^{s,p}_{\mathrm{ball}}(\Omega)} \\
&\lesssim |x - y|^{s-n/p}\|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)} + |x - y|^{s-n/p}[(\delta_1)^{-s+n/p} + (\delta_1)^{n/p}][u(x) - u(y)],
\end{aligned}$$

which together with $s - n/p > 0$ implies that there exists a positive constant $\delta_2 \in (0, \delta_1)$ such that (4.1) holds when $|x - y| \leq \delta_2$.

(iv)$\Rightarrow$(v). Let $u \in \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$ and $g \in \mathcal{D}^s_{\mathrm{ball}}(u)$ with $\|g\|_{L^p(\Omega)} \lesssim \|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)}$. Let $x, y \in \Omega$ be a pair of points satisfying $|x - y| \leq \delta_2$. By an argument similar to the above, we may assume that $0 \leq u(y) < u(x)$ and $u(y) \leq u(z) \leq u(x)$ for all $z \in \Omega$.

Let $N \geq 2$ and φ be a smooth function satisfying that $\varphi(z) = 1$ for $z \in B(x, |x - y|)$, $\operatorname{supp}\varphi \subset B(x, N|x - y|)$, $0 \leq \varphi(z) \leq 1$ and $|\nabla\varphi(z)| \leq 10/(N|x - y|)$ for all $z \in \mathbb{R}^n$. Define $v(z) \equiv [u(z) - u(y)]\varphi(z)$ for all $z \in \Omega$. Then it is easy to check that $v \in \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$ and $g\varphi + 10(N|x - y|)^{-s}[u(x) - u(y)]\chi_{\{B(x,\,N|x-y|)\cap\Omega\}} \in \mathcal{D}^s_{\mathrm{ball}}(v)$, which implies that $\|v\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)} \lesssim \|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)} + (N|x - y|)^{-s+n/p}[u(x) - u(y)]$. Thus

$$\begin{aligned}
|u(x) - u(y)| &= |v(x) - v(y)| \\
&\lesssim |x - y|^{s-n/p}\|v\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)} \\
&\lesssim |x - y|^{s-n/p}\left(\int_{B(x,\,N|x-y|)\cap\Omega} [g(z)]^p\, dz\right)^{1/p} + N^{-s+n/p}[u(x) - u(y)],
\end{aligned}$$

from which and $s - n/p > 0$, it follows that there exists a positive constant $N_1 \equiv N$ large enough such that if $|x - y| \leq \delta_3 \equiv \delta_2$, then (4.2) holds.

(v)$\Rightarrow$(vi). We first prove that Ω is regular. For fixed $x_0 \in \overline{\Omega}$ and $0 < r < \delta_3$, we define $u(z) \equiv \frac{1}{r}d(z, B(x_0, r)^{\complement})$ and $g(z) \equiv r^{-s}\chi_{\Omega\cap\overline{B(x_0,r)}}(z)$ for all $z \in \Omega$. Then similarly to the proof of Theorem 1.1, it is easy to check that there exists a positive constant C independent

of u, x_0, r such that Cg is an element of $\mathcal{D}^s_{\mathrm{ball}}(u)$, which implies that $u \in \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$ and $\|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)} \lesssim r^{-s}|\Omega \cap B(x_0, r)|^{1/p}$. By this and (4.2), we further have

$$1 \le r^{s-n/p} r^{-s} |B(x_0, r) \cap \Omega|^{1/p}$$

and thus $|B(x_0, r) \cap \Omega| \ge r^n$.

Now, let $u \in \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$ and $g \in \mathcal{D}^s_{\mathrm{ball}}(u)$ with $\|g\|_{L^p(\Omega)} \lesssim \|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)}$. Let x, $y \in \Omega$ be a pair of points satisfying $|x - y| < \delta_3/10$. Since Ω is regular, we then have

$$
\begin{aligned}
|u(x) - u_{B(x, 2|x-y|) \cap \Omega}| &\le \sum_{j=-1}^{\infty} |u_{B(x, 2^{-j}|x-y|) \cap \Omega} - u_{B(x, 2^{-j-1}|x-y|) \cap \Omega}| \\
&\lesssim \sum_{j=-1}^{\infty} (2^{-j}|x-y|)^{s-n/p} \left(\fint_{B(x, 2^{-j}N_1|x-y|) \cap \Omega} [g(z)]^p \, dz \right)^{1/p} \\
&\lesssim |x-y|^{s-n/p} \left\{ \mathcal{M}^{\Omega}_{2N_1|x-y|}(g^p) \right\}^{1/p}(x).
\end{aligned}
$$

Similarly, we can prove that

$$|u(y) - u_{B(x, 2|x-y|) \cap \Omega}| \lesssim |x-y|^{s-n/p} \left\{ \mathcal{M}^{\Omega}_{2N_1|x-y|}(g^p) \right\}^{1/p}(y).$$

Thus we obtain (4.3) with $N_2 \equiv 2N_1$ and $\delta_4 \equiv \delta_3/10$ and hence (vi).

(vi)$\Rightarrow$(iv). Let $u \in \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$ and $g \in \mathcal{D}^s_{\mathrm{ball}}(u)$ with $\|g\|_{L^p(\Omega)} \lesssim \|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)}$. By a slight modification of the proof of [11, Theorem 9.5], we know that (4.3) implies that for all $r \le \delta_4/2$,

$$\fint_{B(x, r) \cap \Omega} |u(z) - u_{B(x, r) \cap \Omega}| \, dz \lesssim r^s \left(\fint_{B(x, 6Nr) \cap \Omega} [g(z)]^p \, dz \right)^{1/p},$$

where N is a positive constant independent of u, g and r. By this and the fact that Ω is regular, for $|x - y| \le \delta_4/4$, we have

$$
\begin{aligned}
|u(x) - u_{B(x, 2|x-y|) \cap \Omega}| &\le \sum_{j=-1}^{\infty} |u_{B(x, 2^{-j}|x-y|) \cap \Omega} - u_{B(x, 2^{-j-1}|x-y|) \cap \Omega}| \\
&\lesssim \sum_{j=-1}^{\infty} (2^{-j}|x-y|)^{s-n/p} \left(\int_{B(x, 6 \cdot 2^{-j}N|x-y|) \cap \Omega} [g(z)]^p \, dz \right)^{1/p} \\
&\lesssim |x-y|^{s-n/p} \left(\int_{\Omega} [g(z)]^p \, dz \right)^{1/p},
\end{aligned}
$$

which also holds for $|u(y) - u_{B(x, 2|x-y|) \cap \Omega}|$ by a similar argument. Thus we have (4.1) and (iv).

(iv)$\Rightarrow$(i). By an argument similar to that used in (v)$\Rightarrow$(vi), we know that Ω is regular. Let $u \in M^{s,p}_{\mathrm{ball}}(\Omega)$. Then we only need to prove that $\|u\|_{L^{\infty}(\Omega)} \lesssim \|u\|_{M^{s,p}_{\mathrm{ball}}(\Omega)}$ and for

almost all x, $y \in \Omega$, $|u(x) - u(y)| \lesssim \|u\|_{M_{\mathrm{ball}}^{s,p}(\Omega)}$. In fact, for almost $x \in \Omega$, by the Hölder inequality and (4.1), we have

$$|u(x)| \leq |u(x) - u_{B(x,\delta_2/2) \cap \Omega}| + \|u\|_{L^p(\Omega)} \lesssim \|u\|_{M_{\mathrm{ball}}^{s,p}(\Omega)},$$

which implies that $\|u\|_{L^\infty(\Omega)} \lesssim \|u\|_{M_{\mathrm{ball}}^{s,p}(\Omega)}$. Moreover, for almost all x, $y \in \Omega$, if $|x - y| \geq \delta_2/2$, then $|u(x) - u(y)| \lesssim \|u\|_{L^\infty(\Omega)} \lesssim \|u\|_{M_{\mathrm{ball}}^{s,p}(\Omega)}$; if $|x - y| < \delta_2/2$, then (4.1) yields that $|u(x) - u(y)| \lesssim \|u\|_{M_{\mathrm{ball}}^{s,p}(\Omega)}$. This shows (i) and hence finishes the proof of Theorem 4.2. $\qquad\square$

Remark 4.1. We point out that if Ω is a bounded $\dot{M}_{\mathrm{ball}}^{s,p}$-imbedding domain, then (iv) holds with $\delta_2 = \operatorname{diam}\Omega$ and hence, by the proofs of (iv)$\Rightarrow$(v) and (iv)$\Rightarrow$(v), we can further take $\delta_3 = \operatorname{diam}\Omega$ in (v) and also $\delta_4 = \operatorname{diam}\Omega$ in (vi).

By an argument similar to the proofs of (iv)$\Rightarrow$(v)$\Rightarrow$(vi) in Theorem 4.1 and the observation as in Remark 4.1, we have the following conclusion.

Corollary 4.1. *Let $s \in (0, 1]$ and p, $\widetilde{p} \in (n/s, \infty)$ with $p < \widetilde{p}$. Assume that there exists a positive constant C such that for all $u \in \dot{M}_{\mathrm{ball}}^{s,\widetilde{p}}(\Omega)$, (4.1) of Theorem 4.1 holds with the same constants, namely, for almost all x, $y \in \Omega$ with $|x - y| \leq \delta_2$,*

$$|u(x) - u(y)| \leq C|x - y|^{s - n/p}\|u\|_{\dot{M}_{\mathrm{ball}}^{s,p}(\Omega)}.$$

Then for all $u \in \dot{M}_{\mathrm{ball}}^{s,\widetilde{p}}(\Omega)$ and $g \in \mathcal{D}_{\mathrm{ball}}^s(u)$, (4.2) and (4.3) of Theorem 4.1 still hold with the same constants. Moreover, if Ω is a bounded domain, then (4.2) and (4.3) holds with $\delta_3 = \delta_4 = \operatorname{diam}\Omega$.

The following conclusion clarifies the relations between Hajłasz-Sobolev extensions and imbeddings to some extent, and hence generalizes [19, Theorem A].

Theorem 4.2. *If Ω is an $M_{\mathrm{ball}}^{s,p}$-imbedding (resp. a bounded $\dot{M}_{\mathrm{ball}}^{s,p}$-imbedding) domain for some $s \in (0, 1]$ and $p \in (n/s, \infty)$, then it is an $M_{\mathrm{ball}}^{t,q}$-extension (resp. $\dot{M}_{\mathrm{ball}}^{t,q}$-extension) domain for all $t \in [s, 1]$ and $q \in (n/t, \infty)$ satisfying $t - n/q > s - n/p$.*

To prove Theorem 4.2, we need the following conclusion, which is essentially established in [14] and also [13, 28].

Lemma 4.1. *Let $s \in (0, 1]$ and $p \in (1, \infty)$. A (bounded) domain $\Omega \subset \mathbb{R}^n$ is an $M^{s,p}$-extension (resp. $\dot{M}^{s,p}$-extension) domain if and only if Ω is regular. A (bounded) domain $\Omega \subset \mathbb{R}^n$ is an $M_{\mathrm{ball}}^{s,p}$-extension (resp. $\dot{M}_{\mathrm{ball}}^{s,p}$-extension) domain if and only if Ω is regular and $\dot{M}_{\mathrm{ball}}^{s,p}(\Omega) = \dot{M}^{s,p}(\Omega)$. Moreover, the extension operators can be assumed to be linear.*

Notice that for every $s \in (0, 1]$, $(\mathbb{R}^n, d_s, dx)$ is an Ahlfors n/s-regular metric measure space, where $d_s(x, y) = |x - y|^s$ and dx denotes the Lebesgue measure. Since $M^{s,p}(\Omega) = M^{1,p}(\Omega, d_s, dx)$, Lemma 4.1 for the inhomogeneous Hajłasz-Sobolev spaces is given by [14, Theorem 5] (see also [28, Theorem 1.3]). When Ω is bounded, by an argument similar to that of [14, Theorem 5] (and also [28, Theorem 1.3]), Lemma 4.1 still holds for the homogeneous Hajłasz-Sobolev spaces.

Proof of Theorem 4.2. First, we point out that it suffices to prove Theorem 4.2 for inhomogeneous Hajłasz-Sobolev spaces. To see this, assume that Ω is a bounded $\dot{M}^{s,p}_{\mathrm{ball}}$-imbedding domain. By Theorem 4.1, then Ω is an $M^{s,p}_{\mathrm{ball}}$-imbedding domain. If Theorem 4.2 holds for the inhomogeneous Hajłasz-Sobolev spaces, then for all $t \in [s, 1]$ and $q \in (n/t, \infty)$ satisfying $t - n/q > p - n/s$, Ω is an $M^{t,q}_{\mathrm{ball}}$-extension domain and hence an $M^{t,q}_{\mathrm{ball}}$-imbedding domain, which together with Theorem 4.1 again implies that Ω is an $\dot{M}^{t,q}_{\mathrm{ball}}$-imbedding domain. So, it further suffices to show that a bounded $\dot{M}^{s,p}_{\mathrm{ball}}$-imbedding domain is an $\dot{M}^{s,q}_{\mathrm{ball}}$-extension domain for all $q \in (p, \infty)$. To this end, let $u \in M^{s,q}_{\mathrm{ball}}(\Omega)$ and $g \in \mathcal{D}^s_{\mathrm{ball}}(u)$ with $\|g\|_{L^q(\Omega)} \lesssim \|u\|_{\dot{M}^{s,q}_{\mathrm{ball}}(\Omega)}$. By Corollary 4.1, we know that (4.3) holds for all $x, y \in \Omega$, which means that $\{\mathcal{M}^\Omega_{N_2\delta_4}(g^p)\}^{1/p}$ is a constant multiple of an element of $\mathcal{D}^s(u)$. Hence by the $L^{q/p}(\Omega)$-boundedness of $\mathcal{M}^\Omega_{N_2\delta_4}$, we have $u \in M^{s,q}(\Omega)$ and

$$\|u\|_{\dot{M}^{s,q}(\Omega)} \lesssim \|\{\mathcal{M}^\Omega_{N_2\delta_4}(g^p)\}^{1/p}\|_{L^q(\Omega)} \lesssim \|g\|_{L^q(\Omega)} \lesssim \|u\|_{\dot{M}^{s,q}_{\mathrm{ball}}(\Omega)}.$$

By Lemma 4.1, we deduce that Ω is an $\dot{M}^{s,q}_{\mathrm{ball}}$-extension domain for all $q > p$.

To prove Theorem 4.2 for the inhomogeneous Hajłasz-Sobolev spaces, assume that Ω is an $M^{s,p}_{\mathrm{ball}}$-imbedding domain. Notice that, by Theorem 4.1, Ω is regular. So by Lemma 4.1, it suffices to prove $M^{t,q}_{\mathrm{ball}}(\Omega) = M^{t,q}(\Omega)$. Obviously, $M^{t,q}(\Omega) \subset M^{t,q}_{\mathrm{ball}}(\Omega)$, so we only need to prove that $M^{t,q}_{\mathrm{ball}}(\Omega) \subset M^{t,q}(\Omega)$. We consider the following two cases.

Case $t = s$. Let $u \in M^{s,q}_{\mathrm{ball}}(\Omega)$ and $g \in \mathcal{D}^s_{\mathrm{ball}}(u)$ with $\|g\|_{L^q(\Omega)} \lesssim \|u\|_{\dot{M}^{s,q}_{\mathrm{ball}}(\Omega)}$. Let $\delta \equiv \min\{\delta_3, \delta_4\}$ and $N \equiv \max\{N_1, N_2\}$, where $N_1, N_2, \delta_1, \delta_2$ are as in Theorem 4.1. Then for all $x, y \in \Omega$ with $|x - y| \le \delta$, by Corollary 4.1, we know that (4.3) holds. Notice that by Theorem 4.1, Ω is regular, and by Corollary 4.1 again, (4.2) also holds. So for almost all $x \in \Omega$, we have

$$|u(x)| \le |u(x) - u_{B(x,\delta)\cap\Omega}| + |u_{B(x,\delta)\cap\Omega}|$$
$$\lesssim \fint_{B(x,N\delta)\cap\Omega} |g(z)| \, dz + \fint_{B(x,\delta)\cap\Omega} |u(z)| \, dz$$
$$\lesssim [\mathcal{M}^\Omega_{N\delta}(g^p)(x)]^{1/p} + \mathcal{M}^\Omega_{N\delta}(u)(x).$$

which implies that, for all $x, y \in \Omega$ with $|x - y| \ge \delta$,

$$|u(x) - u(y)| \lesssim |x - y|^s \{\mathcal{M}^\Omega_{N\delta}(u)(x) + [\mathcal{M}^\Omega_{N\delta}(g^p)(x)]^{1/p} + \mathcal{M}^\Omega_{N\delta}(u)(y) + [\mathcal{M}^\Omega_{N\delta}(g^p)(y)]^{1/p}\}.$$

Therefore, $\mathcal{M}^\Omega_{N\delta}(u) + \{\mathcal{M}^\Omega_{N\delta}(g^p)\}^{1/p}$ is a constant multiple of an element of $\mathcal{D}^s(u)$, which together with $q > p$ and the $L^{q/p}(\mathbb{R}^n)$-boundedness of $\mathcal{M}$ implies that $u \in M^{s,q}(\Omega)$ and $\|u\|_{M^{s,q}(\Omega)} \lesssim \|u\|_{M^{s,q}_{\mathrm{ball}}(\Omega)}$.

Case $t > s$. By the *Case $t = s$* and the conclusion of its proof, it suffices to prove that Ω is an $M^{t,q}_{\mathrm{ball}}$-imbedding domain for all $q \in (n/t, \infty)$ satisfying $q - n/t > p - n/s$. Let $u \in M^{t,q}_{\mathrm{ball}}(\Omega)$ and $g \in \mathcal{D}^t_{\mathrm{ball}}(u)$ with $\|g\|_{L^q(\Omega)} \lesssim \|u\|_{\dot{M}^{t,q}_{\mathrm{ball}}(\Omega)}$. Let $x, y \in \Omega$ with $|x - y| \le \min\{\delta_2, \delta_3\}$, where δ_2 and δ_3 are as in Theorem 4.1. Then by Theorem 4.1, it further suffices to prove that there exists a positive constant C independent of u, x and y

such that

(4.4) $$|u(x) - u(y)| \lesssim |x - y|^{t-n/q} \left(\int_{B(x,\, N|x-y|) \cap \Omega} [g(z)]^q \, dz \right)^{1/q}.$$

Without loss of generality, we assume that $0 \le u(y) < u(x)$. Let φ be a smooth function satisfying that $\varphi(z) = 1$ for $z \in B(x, |x-y|)$, $\operatorname{supp} \varphi \subset B(x, N|x-y|)$, $0 \le \varphi(z) \le 1$ and $|\nabla\varphi(z)| \le 10/(N|x-y|)$ for all $z \in \mathbb{R}^n$. Define $v(z) \equiv [u(z) - u(y)]\varphi(z)$ for all $z \in \Omega$. Then it is easy to see that

$$h \equiv g\varphi + 10(N|x-y|)^{-t}[u(x) - u(y)]\chi_{\{B(x,\, N|x-y|) \cap \Omega\}} \in \mathcal{D}^t_{\text{ball}}(v),$$

which implies that $v \in \dot{M}^{t,\,q}_{\text{ball}}(\Omega)$ and

(4.5) $$\|v\|_{\dot{M}^{t,\,q}_{\text{ball}}(\Omega)} \lesssim \|h\|_{L^q(\Omega)} \lesssim \|g\varphi\|_{L^q(\Omega)} + (N|x-y|)^{-t+n/q}[u(x) - u(y)].$$

Moreover, we claim that $\widetilde{h} \equiv \mathcal{M}^{(t-s)}(h)$ is a constant multiple of an element of $\mathcal{D}^{s,\,1/8}_{\text{ball}}(v)$, where

$$\mathcal{M}^{(t-s)}(h)(z) \equiv \sup_{0 < \widetilde{r} < d(z,\, \Omega^\complement)/2} (\widetilde{r})^{t-s} \fint_{B(z,\widetilde{r})} |h(w)| \, dw.$$

In fact, for every pair of points $z, w \in \Omega$ and $0 < |z - w| < \frac{1}{8}d(z, \partial\Omega)$,

$$\begin{aligned}
|v(z) - v_{B(z,\, 2|z-w|)}| &= \sum_{j=-1}^{\infty} |v_{B(z,\, 2^{-j}|z-w|)} - v_{B(z,\, 2^{-j-1}|z-w|)}| \\
&\lesssim \sum_{j=-1}^{\infty} (2^{-j}|z-w|)^{t-n} \int_{B(z,\, 2^{-j}|z-w|)} |h(\widetilde{z})| \, d\widetilde{z} \\
&\lesssim |z - w|^s \mathcal{M}^{(t-s)}(h)(z),
\end{aligned}$$

and similarly, $|v(w) - v_{B(z,\, 2|z-w|)}| \lesssim |z - w|^s \mathcal{M}^{(t-s)}(h)(z)$, which imply the above claim. Let $\widetilde{q} \in (p, \infty)$ such that $1/q - 1/\widetilde{q} = (t-s)/n$. Then by the boundedness from $L^q(\mathbb{R}^n)$ to $L^{\widetilde{q}}(\mathbb{R}^n)$ of $\mathcal{M}^{(t-s)}$ (see [32]), we have $\|\mathcal{M}^{(t-s)}(h)\|_{L^{\widetilde{q}}(\Omega)} \lesssim \|h\|_{L^q(\Omega)}$, which together with Lemma 2.1 implies that $v \in \dot{M}^{s,\,\widetilde{q}}_{\text{ball}}(\Omega)$ and $\|v\|_{\dot{M}^{s,\,\widetilde{q}}_{\text{ball}}(\Omega)} \lesssim \|h\|_{L^q(\Omega)}$. Since $\widetilde{q} > p$, then by *Case $t = s$*, Ω is an $M^{s,\,\widetilde{q}}_{\text{ball}}$-imbedding domain. Then by Theorem 4.1(iv), $\|v\|_{M^{s,\,\widetilde{q}}_{\text{ball}}(\Omega)} \lesssim \|h\|_{L^q(\Omega)}$, (4.5) and $1/q - 1/\widetilde{q} = (t-s)/n$, we have

$$|u(x) - u(y)| = |v(x) - v(y)| \lesssim |x - y|^{s-n/\widetilde{q}} \|v\|_{\dot{M}^{s,\,\widetilde{q}}_{\text{ball}}(\Omega)}$$

$$\lesssim |x - y|^{t-n/q} \left(\int_{B(x,\, N|x-y|) \cap \Omega} [g(z)]^q \, dz \right)^{1/q} + N^{-t+n/q}[u(x) - u(y)].$$

If N is large enough, by $t - n/q > 0$, we then have (4.4), which completes the proof of Theorem 4.2. $\square$

Remark 4.2. (i) Recall that if Ω is an $M^{s,p}_{\text{ball}}$-extension (resp. a bounded $\dot{M}^{s,p}_{\text{ball}}$-extension) domain for some $s \in (0,1]$ and $p \in (n/s, \infty)$, then it is an $M^{s,p}_{\text{ball}}$-imbedding (resp. $\dot{M}^{s,p}_{\text{ball}}$-imbedding) domain, and hence by Theorem 4.2, Ω is an $M^{t,q}_{\text{ball}}$-extension/-imbedding (resp. $\dot{M}^{t,q}_{\text{ball}}$-extension/-imbedding) domain for all $t \in [s,1]$ and $q \in (n/t, \infty)$ satisfying $t - n/q > s - n/p$.

(ii) We also point out that the extension operators in Theorem 4.2 can be assumed to be linear due to Lemma 4.1 below.

By checking the *Case $t = s$* in the proof of Theorem 4.2, we obtain a weak version of Theorem 4.2 when $t = s$ with the aid of Corollary 4.1, which reduces the assumption that Ω is an $\dot{M}^{s,p}$-imbedding appeared in Theorem 4.2 slightly and plays an important role in the proof of Theorem 1.2.

Corollary 4.2. *Let $s \in (0,1]$, $p, \widetilde{p} \in (n/s, \infty)$ with $p < \widetilde{p}$ and Ω be a bounded domain. Suppose that there exists a positive constant C such that if $u \in \dot{M}^{s,\widetilde{p}}_{\text{ball}}(\Omega)$, then $u \in \dot{M}^{s-n/p,\infty}(\Omega)$ and $\|u\|_{\dot{M}^{s-n/p,\infty}(\Omega)} \leq C\|u\|_{\dot{M}^{s,p}_{\text{ball}}(\Omega)}$. Then Ω is an $\dot{M}^{s,\widetilde{p}}_{\text{ball}}$-extension domain.*

The following weak self-improving property established by Shvartsman [31, Theorem 1.4] plays an crucial role in the proof of Theorem 1.2.

Proposition 4.1. *Let $\alpha \in (0,1)$ and Ω be a weak α-cigar domain in $\mathbb{R}^n$. There exist a constant $\alpha^* \in (0, \alpha)$, and constants $\theta, C > 0$ such that the following are true:*

For every $\epsilon > 0$ and $x, y \in \Omega$ with $|x - y| \leq \theta$, there exist a rectifiable curve $\Gamma \subset \Omega$ joining x and y and a subset $\widetilde{\Gamma} \subset \Gamma$ consisting of a finite number of arcs such that the following conditions are satisfied:

(i) For every $\tau \in [\alpha^*, \alpha]$,

$$(4.6) \qquad \int_{\widetilde{\Gamma}} [d(z, \Omega^{\complement})]^{\tau - 1} \, |dz| \leq C|x - y|^{\tau}.$$

In addition, for every ball B centered in $\widetilde{\Gamma}$ of radius at most $|x - y|$, $\operatorname{diam} B \leq C\ell(B \cap \widetilde{\Gamma})$.
(ii) We have $\ell(\Gamma) \leq C|x - y|$ and

$$(4.7) \qquad \ell(\Gamma \setminus \widetilde{\Gamma}) < \epsilon.$$

Moreover,

$$(4.8) \qquad \int_{\Gamma \setminus \widetilde{\Gamma}} [d(z, \Omega^{\complement})]^{\alpha - 1} \, |dz| \leq C|x - y|^{\alpha}.$$

Proof of Theorem 1.2. (i) Assume that Ω is a weak α-cigar domain with $\alpha \in (0,1)$. Let $s \in (0,1]$ and $\widetilde{p} \equiv (n - \alpha)/(s - \alpha)$ be fixed. Let $\alpha^* \in (0, \alpha)$, θ and C_1 be as in Proposition 4.1, and also $p^* \equiv (n - \alpha^*)/(s - \alpha^*)$. Then $\alpha^* = (p^*s - n)/(p^* - 1)$, $\alpha = (\widetilde{p}s - n)/(\widetilde{p} - 1)$ and $n/s < p^* < \widetilde{p} < \infty$. Moreover, without loss of generality, we may assume that $\theta = \operatorname{diam}\Omega$.

We first claim that for every $p \in [p^*, \widetilde{p}]$, there exists a positive constant C such that for all $u \in M^{s,\widetilde{p}}_{\text{ball}}(\Omega)$, $u \in \dot{M}^{s-n/p,\infty}(\Omega)$ and

$$(4.9) \qquad \|u\|_{\dot{M}^{s-n/p,\infty}(\Omega)} \leq C\|u\|_{\dot{M}^{s,p}_{\text{ball}}(\Omega)}.$$

Then Theorem 1.2(i) then follows from this claim. Indeed, the above claim implies that Ω is an $\dot{M}^{s,\widetilde{p}}_{\text{ball}}$-imbedding domain, which together with Theorem 4.2 further yields that Ω is an $\dot{M}^{s,p}_{\text{ball}}$-extension and $\dot{M}^{s,p}_{\text{ball}}$-imbedding domain for all $p \in (\widetilde{p}, \infty)$. Moreover, the above claim together with Corollary 4.2 implies that Ω is an $\dot{M}^{s,\widetilde{p}}_{\text{ball}}$-extension domain.

Now we turn to the proof of the above claim. Without loss of generality, we may also assume that $\|u\|_{\dot{M}^{s,p}_{\text{ball}}(\Omega)} = 1$. Then the proof of (4.9) is reduced to checking that for all Lebesgue points $x, y \in \Omega$ of u,

$$(4.10) \qquad |u(x) - u(y)| \lesssim |x-y|^{s-n/p}.$$

To this end, take $g \in \mathcal{D}^s(u) \cap L^p(\Omega) \cap L^{\widetilde{p}}(\Omega)$ such that $1 \le \|g\|_{L^p(\Omega)} \le 2$. If $|x-y| \le d(x, \Omega^\complement)/4$, then

$$(4.11) \quad |u(x) - u_{B(x, 2|x-y|)}| \le \sum_{j=0}^{\infty} |u_{B(x, 2^{-j}|x-y|)} - u_{B(x, 2^{-j+1}|x-y|)}|$$
$$\lesssim \sum_{j=0}^{\infty} \fint_{B(x, 2^{-j}|x-y|)} |u(z) - u_{B(x, 2^{-j}|x-y|)}| \, dz$$
$$\lesssim \sum_{j=-1}^{\infty} [2^{-j}|x-y|]^s \fint_{B(x, 2^{-j}|x-y|)} g(z) \, dz$$
$$\lesssim \sum_{j=-1}^{\infty} [2^{-j}|x-y|]^{s-n/p} \left\{ \int_{B(x, 2^{-j}|x-y|)} [g(z)]^p \, dz \right\}^{1/p}$$
$$\lesssim |x-y|^{s-n/p} \left\{ \int_{B(x, 2|x-y|)} [g(z)]^p \, dz \right\}^{1/p},$$

which together with $\|g\|_{L^p(\Omega)} \sim 1$ gives that $|u(x) - u_{B(x, 2|x-y|)}| \lesssim |x-y|^{s-n/p}$. Similarly, we can prove that $|u(y) - u_{B(x, 2|x-y|)}| \lesssim |x-y|^{s-n/p}$. Thus,

$$|u(x) - u(y)| \le |u(x) - u_{B(x, 2|x-y|)}| + |u(y) - u_{B(x, 2|x-y|)}| \lesssim |x-y|^{s-n/p},$$

which gives (4.10) when $|x-y| \le d(x, \Omega^\complement)/4$.

Now we assume that $|x-y| > \max\{d(x, \Omega^\complement), d(y, \Omega^\complement)\}/4$. Then for $\epsilon > 0$ fixed, let Γ joining x and y, and a subset $\widetilde{\Gamma}$ of Γ be as in Proposition 4.1. Then by (4.6) and (4.8), we have
$$\int_{\Gamma} [d(z, \Omega^\complement)]^{(\widetilde{p}s-n)/(\widetilde{p}-1)-1} |dz| \lesssim |x-y|^{(\widetilde{p}s-n)/(\widetilde{p}-1)}.$$

By using the Bescovitch covering lemma (see [32]) and some arguments similar to these in the proofs of [2, Theorem 4.1] and [31, Lemma 3.2], we can find a family of balls $\mathcal{B} \equiv \{B_i\}_{i=1}^N$ such that
   a) $B_i \equiv B(z_i, d(z_i, \Omega^\complement)/50)$ with $z_i \in \Gamma$ for all $i = 0, \cdots, N$, $z_0 = x$ and $z_N = y$;
   b) $B_i \cap B_{i+1} \ne \emptyset$ for all $i = 0, \cdots, N-1$;

c) $\sum_{i=1}^{N} \chi_{2B_i}(z) \leq C_2$ for all $z \in \Omega$, where the constant C_2 only depends on the dimension n.

Let $w_i \in B_i \cap B_{i+1}$ for all $i = 0, \cdots, N-1$. Then by the properties a) and b) above, (4.11) and an argument similar to (4.11), we have

$$
\begin{aligned}
|u(x) - u(y)| &\leq \sum_{i=0}^{N-1} (|u(z_i) - u(w_i)| + |u(w_i) - u(z_{i+1})|) \\
&\leq \sum_{i=0}^{N-1} (|u(z_i) - u_{B(z_i,\, 2|w_i-z_i|)}| + |u(w_i) - u_{B(z_i,\, 2|w_i-z_i|)}| \\
&\quad + |u(w_i) - u_{B(z_{i+1},\, 2|w_i-z_{i+1}|)}| + |u(z_{i+1}) - u_{B(z_{i+1},\, 2|w_i-z_{i+1}|)}|) \\
&\lesssim \sum_{i=0}^{N} [d(z_i, \Omega^{\complement})]^{s-n/p} \left(\fint_{2B_i} [g(z)]^p \, dz \right)^{1/p}.
\end{aligned}
$$

Let $\widetilde{\mathcal{B}}$ be the collection of all $B_i \in \mathcal{B}$ such that $B_i \cap \widetilde{\Gamma} \neq \emptyset$. Then by the Hölder inequality, we have

$$
\begin{aligned}
|u(x) - u(y)| &\lesssim \sum_{B_i \in \widetilde{\mathcal{B}}} [d(z_i, \Omega^{\complement})]^{s-n/p} \left(\fint_{2B_i} [g(z)]^p \, dz \right)^{1/p} \\
&\quad + \sum_{B_i \in \mathcal{B} \setminus \widetilde{\mathcal{B}}} [d(z_i, \Omega^{\complement})]^{s-n/\widetilde{p}} \left(\fint_{2B_i} [g(z)]^{\widetilde{p}} \, dz \right)^{1/\widetilde{p}} \\
&\lesssim \left(\sum_{B_i \in \widetilde{\mathcal{B}}} [d(z_i, \Omega^{\complement})]^{(ps-n)/(p-1)} \right)^{(p-1)/p} \left(\sum_{B_i \in \mathcal{B}} \fint_{2B_i} [g(z)]^p \, dz \right)^{1/p} \\
&\quad + \left(\sum_{i=0}^{N} [d(z_i, \Omega^{\complement})]^{(\widetilde{p}s-n)/(\widetilde{p}-1)} \right)^{(\widetilde{p}-1)/\widetilde{p}} \left(\sum_{B_i \in \mathcal{B} \setminus \widetilde{\mathcal{B}}} \fint_{2B_i} [g(z)]^{\widetilde{p}} \, dz \right)^{1/\widetilde{p}} \\
&\equiv I_1 + I_2.
\end{aligned}
$$

To estimate I_1, for each $B_i \in \widetilde{\mathcal{B}}$, take $\widetilde{z}_i \in B_i \cap \widetilde{\Gamma}$. It is easy to see that

$$
\frac{24}{25} d(z_i, \Omega^{\complement}) \leq d(z, \Omega^{\complement}) \leq \frac{25}{24} d(z_i, \Omega^{\complement}),
$$

for all $z \in 2B_i$, especially, for $z = \widetilde{z}_i$. By this and Proposition 4.1, $d(\widetilde{z}_i, \Omega^{\complement}) \lesssim \ell(\widetilde{\Gamma} \cap 2B_i)$. Thus,

$$
[d(z_i, \Omega^{\complement})]^{(ps-n)/(p-1)} \lesssim \int_{\widetilde{\Gamma} \cap 2B_i} [d(z, \Omega^{\complement})]^{(ps-n)/(p-1)-1} |dz|,
$$

which together with $\sum_{B_i} \chi_{2B_i} \lesssim 1$ and (4.6) of Proposition 4.1 with $\tau \equiv (ps-n)/(p-1) \in [\alpha^*, \alpha]$ implies that

$$
I_1 \lesssim \left(\sum_{B_i \in \widetilde{\mathcal{B}}} \int_{\widetilde{\Gamma} \cap 2B_i} [d(z, \Omega^{\complement})]^{(ps-n)/(p-1)-1} |dz| \right)^{(p-1)/p} \left(\sum_{B_i \in \mathcal{B}} \fint_{2B_i} [g(z)]^p \, dz \right)^{1/p}
$$

$$\lesssim \left(\int_{\widetilde{\Gamma}} [d(z,\, \Omega^{\complement})]^{(ps-n)/(p-1)-1} |dz| \right)^{(p-1)/p} \left(\int_{\Omega} [g(z)]^p \, dz \right)^{1/p}$$

$$\lesssim |x-y|^{s-n/p}.$$

By an argument as in the estimate of I_1, we then have

$$I_2 \lesssim \left(\sum_{B_i \in \mathcal{B}} \int_{\Gamma \cap 2B_i} [d(z,\, \Omega^{\complement})]^{(\widetilde{p}s-n)/(\widetilde{p}-1)-1} |dz| \right)^{(\widetilde{p}-1)/\widetilde{p}} \left(\sum_{B_i \in \mathcal{B} \setminus \widetilde{\mathcal{B}}} \int_{2B_i} [g(z)]^{\widetilde{p}} \, dz \right)^{1/\widetilde{p}}$$

$$\lesssim |x-y|^{s-n/\widetilde{p}} \left(\int_{\cup_{B_i \in \mathcal{B} \setminus \widetilde{\mathcal{B}}} 2B_i} [g(z)]^{\widetilde{p}} \, dz \right)^{1/\widetilde{p}}.$$

Moreover, notice that if $B_i \in \mathcal{B} \setminus \widetilde{\mathcal{B}}$, then $B_i \cap \Gamma \subset \Gamma \setminus \widetilde{\Gamma}$, which together with (4.7) implies that

$$\sum_{B_i \in \mathcal{B} \setminus \widetilde{\mathcal{B}}} |B_i|^{1/n} \leq \sum_{B_i \in \mathcal{B} \setminus \widetilde{\mathcal{B}}} \int_{\Gamma \setminus \widetilde{\Gamma}} \chi_{B_i}(z) \, |dz| \leq \int_{\Gamma \setminus \widetilde{\Gamma}} \sum_{B_i \in \mathcal{B}} \chi_{B_i}(z) \, |dz| \lesssim \ell(\Gamma \setminus \widetilde{\Gamma}) \lesssim \epsilon$$

and hence

$$|\cup_{B_i \in \mathcal{B} \setminus \widetilde{\mathcal{B}}} 2B_i| \lesssim \sum_{B_i \in \mathcal{B} \setminus \widetilde{\mathcal{B}}} |B_i| \lesssim \left(\sum_{B_i \in \mathcal{B} \setminus \widetilde{\mathcal{B}}} |B_i|^{1/n} \right)^{1/n} \lesssim \epsilon.$$

Since the constants that appeared in the estimates of I_1 and I_2 are independent of ϵ, by $g \in L^p(\Omega)$ and absolute continuity of integral with respect to measure, we can choose ϵ small enough such that $I_2 \leq |x-y|^{s-n/p}$. Combining the estimates of I_1 and I_2, we obtain (4.10) for almost all $x,\, y \in \Omega$. This finishes the proof of Theorem 1.2(i).

(ii) Assume that Ω has the slice property as in Definition 2.4 with constant C_0 and also that Ω is a bounded $\dot{M}_{\text{ball}}^{s,p}$-imbedding domain. Observe that, by [2], a bounded weak α-cigar domain with $\alpha \in (0,\, 1)$ is weak β-cigar domain for all $\beta \in (\alpha,\, 1]$. So it suffices to prove that Ω is a weak $(ps-n)/(p-1)$-cigar domain. Without loss of generality, we may assume that $j \geq 2$ in Definition 2.4. Indeed, if $j = 0$, then necessarily $x = y$. If $j = 1$, by Definition 2.4(iii) and (iv), we have that $d(z,\, \Omega^{\complement}) \gtrsim d(x,\, \Omega^{\complement})$ for all $z \in S_0 \cap \gamma$ and $d(z,\, \Omega^{\complement}) \gtrsim d(y,\, \Omega^{\complement})$ for all $z \in S_1 \cap \gamma$. This together with Definition 2.4(iii) implies that

$$\int_{\gamma} [d(z,\, \Omega^{\complement})]^{(ps-n)/(p-1)-1} \, |dz| \lesssim [d(x,\, \Omega^{\complement})]^{(ps-n)/(p-1)} + [d(y,\, \Omega^{\complement})]^{(ps-n)/(p-1)}$$

$$\lesssim |x-y|^{(ps-n)/(p-1)},$$

as desired.

Suppose thus $x,\, y \in \Omega$ are fixed, and γ and $\{S_i\}_{i=0}^j$ be as in Definition 2.4 with $j \geq 2$. For each $i = 1, \cdots, j-1$, define function u_i by setting $u_i(z) \equiv \inf_{\widetilde{\gamma}} \ell(\widetilde{\gamma} \cap S_i)$ for all $z \in \Omega$, where the infimum is taken over all the rectifiable curves $\widetilde{\gamma}$ joining x and z. Obviously, $u_i(z) = 0$ for $z \in \cup_{k=0}^{i-1} S_k$ and $u_i(z)$ is a constant for $z \in \cup_{k=i+1}^{j} S_k$. Similarly to the

proof of Theorem 1.1, there exists a positive constant $\widetilde{C}$ independent of x, y, i such that $g \equiv \widetilde{C} r_i^{1-s} \chi_{B(x_i, 2C_0 d(x_i, \Omega^\complement))}$ is an element of $\mathcal{D}_{\mathrm{ball}}^{s, 1/(8C_0)}(u)$, where $r_i = \operatorname{diam} S_i \sim d(x_i, \Omega^\complement)$ by Definition 2.4(iii), and which implies that $u \in \dot{M}_{\mathrm{ball}}^{s, p}(\Omega)$ and $\|u_i\|_{\dot{M}_{\mathrm{ball}}^{s, p}(\Omega)} \lesssim r_i^{1-s+n/p}$. Notice also that $|u_i(x) - u_i(y)| \geq \delta_i$.

Moreover, let

$$(4.12) \qquad u \equiv \sum_{i=1}^{j-1} r_i^{(s-n/p)/(p-1)} r_i^{s-1-n/p} u_i$$

and

$$g \equiv \sum_{i=1}^{j-1} r_i^{(s-n/p)/(p-1)} r_i^{s-1-n/p} g_i = \sum_{i=1}^{j-1} r_i^{(s-n/p)/(p-1)} r_i^{-n/p} \chi_{B(x_i, 2C_0 d(x_i, \Omega^\complement))}.$$

Then there exists a positive constant $\widetilde{C}$ independent of x, y such that $\overline{C} g$ is an element of $\mathcal{D}_{\mathrm{ball}}^{s, 1/(8C_0)}(u)$, which together with Definition 2.4 and the vector-valued inequality of the Hardy-Littlewood maximal function $\mathcal{M}$ (see, for example, [32]) implies that $u \in \dot{M}_{\mathrm{ball}}^{s, p}(\Omega)$ and

$$\|u\|_{\dot{M}_{\mathrm{ball}}^{s, p}(\Omega)}^p \lesssim \int_\Omega \left(\sum_{i=1}^{j-1} r_i^{(s-n/p)/(p-1)} r_i^{-n/p} \chi_{B(x_i, 2C_0 d(x_i, \Omega^\complement))}(z) \right)^p dz$$

$$\lesssim \int_\Omega \left\{ \sum_{i=1}^{j-1} \left[\mathcal{M}\left(\left[r_i^{(s-n/p)/(p-1)} r_i^{-n/p} \chi_{B(x_i, C_0^{-1} d(x_i, \Omega^\complement))} \right]^{1/2} \right)(z) \right]^2 \right\}^p dz$$

$$\lesssim \int_\Omega \left(\sum_{i=1}^{j-1} r_i^{(s-n/p)/(p-1)} r_i^{-n/p} \chi_{B(x_i, C_0^{-1} d(x_i, \Omega^\complement))}(z) \right)^p dz$$

$$\lesssim \sum_{i=1}^{j-1} r_i^{p(s-n/p)/(p-1)}.$$

By this and the assumption that Ω is an $\dot{M}_{\mathrm{ball}}^{s, p}$-imbedding domain, we have

$$|u(x) - u(y)| \lesssim |x - y|^{s-n/p} \|u\|_{\dot{M}_{\mathrm{ball}}^{s, p}(\Omega)} \lesssim |x - y|^{s-n/p} \left(\sum_{i=1}^{j-1} r_i^{p(s-n/p)/(p-1)} \right)^{1/p},$$

which together with

$$\sum_{i=1}^{j-1} r_i^{(s-n/p)/(p-1)} r_i^{s-n/p} \leq \sum_{i=1}^{j-1} r_i^{(s-n/p)/(p-1)} r_i^{s-1-n/p} |u_i(x) - u_i(y)| \leq |u(x) - u(y)|$$

implies that

$$\sum_{i=1}^{j-1} r_i^{(ps-n)/(p-1)} \lesssim |x - y|^{(ps-n)/(p-1)}.$$

Thus

$$\int_\gamma [d(z, \Omega^\complement)]^{(ps-n)/(p-1)-1} \, |dz| \lesssim |x - y|^{(ps-n)/(p-1)},$$

which gives Theorem 1.2(ii). This finishes the proof of Theorem 1.2. $\qquad\square$

Remark 4.3. Observe that in the proof of Theorem 1.2(ii), the functions $\{u_i\}_{i=1}^{j-1}$ and hence u defined in (4.12) belong to $\dot{M}_{\mathrm{ball}}^{s,\infty}(\Omega)$. So to obtain that Ω is a weak $(ps-n)/(p-1)$-cigar domain, the assumption that Ω is an $\dot{M}_{\mathrm{ball}}^{s,p}$-imbedding domain required in Theorem 1.2(ii) can be reduced to a weaker one: for all $u \in \dot{M}_{\mathrm{ball}}^{s,\infty}(\Omega)$, $\|u\|_{\dot{M}^{s-n/p,\infty}(\Omega)} \lesssim \|u\|_{\dot{M}_{\mathrm{ball}}^{s,p}(\Omega)}$. This leads to the proof of Theorem 1.3.

Proof of Theorem 1.3. Let Ω be a weak α-cigar domain satisfying the slice property. Let $\alpha^* \in (0, \alpha)$, $s \in (0, 1]$ and $p^*, \widetilde{p} \in (n/s, \infty)$ be as in the proof of Theorem 1.2(i). Then by the conclusion (4.9) there, $\|u\|_{\dot{M}^{s-n/p^*,\infty}(\Omega)} \lesssim \|u\|_{\dot{M}_{\mathrm{ball}}^{s,p^*}(\Omega)}$ for all $u \in M_{\mathrm{ball}}^{s,\widetilde{p}}(\Omega)$ and hence for all $u \in \dot{M}_{\mathrm{ball}}^{s,\infty}(\Omega)$. So keeping the the observation in Remark 4.3 in mind and running the proof of Theorem 1.2(ii) again, we obtain that Ω is a weak α^*-cigar domain. Then by Theorem 1.2(i), we further know that Ω is an $\dot{M}_{\mathrm{ball}}^{t,q}$-extension/-imbedding domain for all $t \in (\alpha^*, 1]$ and $q \in [(n-\alpha^*)/(t-\alpha^*), \infty)$, which completes the proof of Theorem 1.3. $\quad\square$

References

[1] S. M. Buckley and P. Koskela, Sobolev-Poincaré implies John. Math. Res. Lett. 2 (1995), 577-594.

[2] S. M. Buckley and P. Koskela, Criteria for imbeddings of Sobolev-Poincaré type, Internat. Math. Res. Notices 18 (1996), 881-901.

[3] S. M. Buckley and A. Stanoyevitch, Weak slice conditions and Hölder imbeddings, J. London Math. Soc. (2) 64 (2001), 690-706.

[4] F. W. Gehring, Univalent functions and the Schwarzian derivative, Comment. Math. Helv. 52 (1977), 561-572.

[5] F. W. Gehring and O. Martio, Lipschitz classes and quasiconformal mappings, Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985), 203-219.

[6] F. W. Gehring and O. Martio, Quasiextremal distance domains and extension of quasiconformal mappings, J. Analyse Math. 45 (1985), 181-206.

[7] V. M. Gol'dšteĭn, Quasiconformal, quasi-isometric mappings and the Sobolev spaces, Complex analysis and applications '81 (Varna, 1981), 202-212, Publ. House Bulgar. Acad. Sci., Sofia, 1984.

[8] V. M. Gol'dšteĭn and Yu. G. Reshetnyak, Introduction to the theory of functions with generalized derivatives, and quasiconformal mappings (Russian), Moscow, 1983. 285 pp.

[9] V. M. Gol'dšteĭn and S. K. Vodop'anov, Prolongement de fonctions différentiables hors de domaines plans, C. R. Acad. Sci. Paris Sér. I Math. 293 (1981), 581-584.

[10] P. Hajłasz, Sobolev spaces on an arbitrary metric spaces, Potential Anal. 5 (1996), 403-415.

[11] P. Hajłasz, Sobolev spaces on metric-measure spaces. Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 173-218, Contemp. Math., 338, Amer. Math. Soc., Providence, RI, 2004.

[12] P. Hajłasz and P. Koskela, Sobolev met Poincaré, Memoirs Amer. Math. Soc. 145 (688) (2000), 1-101.

[13] P. Hajłasz, P. Koskela and H. Tuominen, Sobolev imbeddings, extensions and measure density condition, J. Funct. Anal. 254 (2008), 1217-1234.

[14] P. Hajłasz, P. Koskela and H. Tuominen, Measure density and extendability of Sobolev functions, Rev. Mat. Iberoam. 24 (2008), 645-669.

[15] P. Hajłasz and O. Martio, Traces of Sobolev functions on fractal type sets and characterization of extension domains, J. Funct. Anal. 143 (1997), 221-246.

[16] J. Heinonen and P. Koskela, Quasiconformal maps in metric spaces with controlled geometry, Acta Math. 181 (1998), 1-61.

[17] P. W. Jones, Quasiconformal mappings and extendability of functions in Sobolev spaces, Acta Math. 147 (1981), 71-88.

[18] P. Koskela, Capacity extension domains, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes No. 73 (1990), 42 pp.

[19] P. Koskela, Extensions and imbeddings, J. Funct. Anal. 159 (1998), 369-384.

[20] P. Koskela and F. Reitich, Hölder continuity of Sobolev functions and quasiconformal mappings, Math. Z. 213 (1993), 457-472.

[21] P. Koskela and E. Saksman, Pointwise characterizations of Hardy-Sobolev functions. Math. Res. Lett. 15 (2008), 727-744.

[22] P. Koskela, D. Yang and Y. Zhou, A characterization of Hajłasz-Sobolev and Triebel-Lizorkin spaces, J. Funct. Anal. 258 (2010), 2637-2661.

[23] P. Koskela, D. Yang and Y. Zhou, Pointwise characterizations of Besov and Triebel-Lizorkin spaces and quasiconformal mappings, submitted.

[24] V. G. Maz'ya, Sobolev Spaces, Translated from the Russian by T. O. Shaposhnikova, Springer-Verlag, Berlin, 1985.

[25] A. Miyachi, Hardy-Sobolev spaces and maximal functions, J. Math. Soc. Japan 42 (1990), 73-90.

[26] V. S. Rychkov, On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains, J. London Math. Soc. (2) 60 (1999), 237-257.

[27] V. S. Rychkov, Linear extension operators for restrictions of function spaces to irregular open sets, Studia Math. 140 (2000), 141-162.

[28] P. Shvartsman, Local approximations and intrinsic characterizations of spaces of smooth functions on regular subsets of $\mathbb{R}^n$, Math. Nachr. 279 (2006), 1212-1241.

[29] P. Shvartsman, On extension of Sobolev functions defined on regular subsets of metric measure spaces, J. Approx. Theory 144 (2007), 139-161.

[30] P. Shvartsman, Sobolev W_p^1-spaces on closed subsets of R^n, Adv. Math. 220 (2009), 1842-1922.

[31] P. Shvartsman, On Sobolev extension domains in R^n, J. Funct. Anal. 258 (2010), 2205-2245.

[32] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Univ. Press, Princeton, N. J., 1993.

[33] H. Triebel, Theory of Function Spaces. Monographs in Mathematics, 78. Birkhäuser Verlag, Basel, 1983.

[34] H. Triebel, Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers, Rev. Mat. Complut. 15 (2002), no. 2, 475-524.

[35] J. Väisälä, Quasi-Möbius maps, J. Analyse Math. 44 (1984/85), 218-234.

[36] S. K. Vodop'janov, V. M. Gol'dšteĭn and T. G. Latfullin, A criterion for the extension of functions of the class L_2^1 from unbounded plane domains, Sibirsk. Mat. Zh. 20 (1979), 416-419.

[37] D. Yang, New characterizations of Hajłasz-Sobolev spaces on metric spaces, Sci. China Ser. A 46 (2003), 675-689.

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